LINEAR RESPONSE FORMULA FOR EQUILIBRIUM STATES IN NON-UNIFORMLY EXPANDING DYNAMICS

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Abstract. In this paper we give further contributions to the ergodic theory of a robust class of local diffeomorphisms with non-uniform expansion and where no Markov assumption is required. We prove that the topological pressure is differentiable as a function of the dynamics and the potential and provide a formula to the differentiable dependence of equilibrium states. Moreover we prove differentiability of the maximal entropy measure and continuity of extremal Lyapunov exponents and metric entropy with respect to the dynamics. Finally we obtain a local large deviation principle for the equilibrium states and show that the rate function is continuous with respect to the dynamics and the potential.

1. Introduction

The thermodynamical formalism was brought from statistical mechanics to dynamical systems by the pioneering works of Sinai, Ruelle and Bowen [Sin72, Bow75, BR75] in the mid seventies. Indeed, the correspondance between one-dimensional lattices and uniformly hyperbolic maps, via Markov partitions, allowed to translate and introduce several notions of Gibbs measures and equilibrium states in the realm of dynamical systems. Nevertheless, although uniformly hyperbolic dynamics arise in physical systems (see e.g. [HM03]) they do not include some relevant classes of systems including the Manneville-Pomeau transformation (phenomena of intermittency), Hénon maps and billiards with convex scatterers. We note that all the previous systems present some non-uniformly hyperbolic behavior and its relevant measure satisfies some weak Gibbs property. More recently there have been established many evidences that non-uniformly hyperbolic dynamical systems admit countable and generating Markov partitions. This is now paralel to the development of a thermodynamical formalism of gases with infinitely many states, a hard subject not yet completely understood. We refer the reader to [Sar99, Pin08, PV10] for some recent progress in this direction.

A cornerstone of the theory that has driven the recent attention of many authors both in the physics and mathematics literature concerns the differentiability of thermodynamical quantities as the topological pressure, SRB measures or equilibrium states with respect to the underlying dynamical system. For natural reasons these have been refered as linear response formulas. Several important results have been obtained for uniformly hyperbolic maps and flows, some partially hyperbolic diffeomorphisms and some one-dimensional piecewise expanding maps and nonuniformly expanding quadratic maps. We refer the reader to [KKPW89, Rue97, Rue04, Dol04, Rue05, BL07, BS08, BS09, Rue09, BS10] for precise statements. Nevertheless, and

Date: March 4, 2013.

despite the effort of many authors, a general picture is still far from complete. On the one hand the study of finer statistical properties of thermodynamical quantities as equilibrium states, mixing properties, large deviation and limit theorems, stability under deterministic perturbations or regularity of the topological pressure is usually associated to good spectral properties of the Ruelle-Perron-Frobenius operator. For that reason the functional analytic approach to thermodynamical formalism has gained special interest in the last few years and produced new and interesting results even in both uniformly and non-uniformly hyperbolic setting (see e.g. [BKL01, Cas02, Cas04, GL06, BT07, DL08, BG09, BG10, Ru10]). On the other hand, neither the stability of the equilibrium states or differentiability results for on thermodynamical quantities could follow directly from the spectral gap property since transfer operators acting on the space of Hölder continuous potentials may not even vary continuously with the dynamical system (see e.g. [CV11, Example 4.14]).

In this article we give first contributions to the study of linear response formulas in a multidimensional nonuniformly hyperbolic context. We study a robust class of multidimensional nonuniformly expanding maps considered in [VV10, CV11] that admit the coexistence of expanding and contracting behavior and need not admit any Markov partition. Such class of maps obtained important examples obtained by bifurcation of expanding homeomorphisms, subshifts of finite type and intermitency phenomena as in the class of Maneville-Pommeau maps. In [CV11] the authors used Birkhoff's method of projective cones aplied to the Ruelle-Perron-Frobenius operator acting on good Banach spaces to establish that the Ruelle-Perron-Frobenius transfer operator has a spectral gap in the Banach spaces of both Hölder continuous and smooth observables, to obtain good statistical properties of equilibrium states and obtain continuous dependence of the topological pressure with respect to the dynamics and the potential. Here we deduce further regularity of the topological pressure and the density of the equilibrium states with respect to conformal measures along parametrized families of potentials (e.g. real analytic) and the study of zeta functions. Such program has been carried out with success for uniformly hyperbolic and some partially hyperbolic and one-dimensional non-uniformly expanding dynamical systems. See e.g. [Rue97, Dol04, BS08, BS09] and references therein. To the best of our knowledge these are the first differentiabiliy formulas for the topological pressure and equilibrium states for multidimensional non-uniformly expanding maps. We also obtain some important large deviation results and deduce that the rate function varies differentiably with respect to the dynamical system and potential. These are new results even in the uniformly hyperbolic setting. We refer the reader to Section 2 for precise statements.

Let us mention that our strategy can probably apply whenever the Ruelle-Perron-Frobenius has a spectral gap property with uniform bounds, independently of its continuous dependence or not in the operator norm.

This paper is organized as follows. In Section 2, we recall some definitions and state our main results. Some preliminary results are given in Section 3. Our main results concerning the differentiability of topological pressure, conformal measures and equilibrium states are proven in Section 4 and we prove our local large deviations principle in Section 5. Finally, some examples are discussed in Section 6.

2. Statement of the main results

2.1. **Setting.** In this section we introduce some definitions and establish the setting. Let M be compact and connected Riemmanian manifold of dimension m with distance d. Let $f : M \to M$ be a *local homeomorphism* and assume that there exists a continuous function $x \mapsto L(x)$ such that, for every $x \in M$ there is a neighborhood U_x of x so that $f_x : U_x \to f(U_x)$ is invertible and

$$d(f_x^{-1}(y), f_x^{-1}(z)) \leq L(x)\ d(y,z), \quad \forall y,z \in f(U_x).$$

In particular every point has the same finite number of preimages $\deg(f)$ which coincides with the degree of f.

For all our results we assume that f satisfies conditions (H1) and (H2) below. Assume there exist constants $\sigma > 1$ and $L \geq 1$, and an open region $\mathcal{A} \subset M$ such that

(H1) $L(x) \leq L$ for every $x \in \mathcal{A}$ and $L(x) < \sigma^{-1}$ for all $x \notin \mathcal{A}$, and L is close to 1: the precise condition is given in (3.1) and (3.2).

(H2) There exists a finite covering $\mathcal{U}$ of M by open domains of injectivity for f such that $\mathcal{A}$ can be covered by $q < \deg(f)$.

The first condition means that we allow expanding and contracting behavior to coexist in M: f is uniformly expanding outside $\mathcal{A}$ and not too contracting inside $\mathcal{A}$. In the case that $\mathcal{A}$ is empty then f is uniformly expanding. The second condition requires that every point has at least one preimage in the expanding region.

An observable $g : M \to \mathbb{R}$ is α-Hölder continuous if the Hölder constant

$$|g|_\alpha = \sup_{x \neq y} \frac{|g(x) - g(y)|}{d(x,y)^\alpha}$$

is finite. As usual, we endow the space $C^\alpha(M, \mathbb{R})$ of Hölder continuous observables with the norm $\| \cdot \|_\alpha = \| \cdot \|_0 + | \cdot |_\alpha$. Throughout, we let $\phi : M \to \mathbb{R}$ denote a potential at least Hölder continuous and satisfying either

(P) $\sup \phi - \inf \phi < \varepsilon_\phi$ and $|e^\phi|_\alpha < \varepsilon_\phi\ e^{\inf \phi}$

provided that ϕ is Hölder continuous, or

(P') $\sup \phi - \inf \phi < \varepsilon_\phi$ and $\max_{s \leq r} \|D^s \phi\|_0 < \varepsilon_\phi$

if ϕ is C^r, where $\varepsilon_\phi > 0$ depends only on L, σ, q, $\deg(f)$, r, a positive integer m and small $\delta > 0$ stated precisely in [CV11] (see equations (3.1) and (3.2) below). This is an open condition on the set of potentials, satisfied by constant potentials. In particular we consider measures of maximal entropy.

2.2. **Strong statistical properties of equilibrium states.** Let us first introduce some necessary definitions and collect from [VV10, CV11] some of the known results on the existence and statistical properties of equilibrium states for this robust class of transformations. Given a continuous map $f : M \to M$ and a potential $\phi : M \to \mathbb{R}$, the variational principle for the pressure asserts that

$$P_{\text{top}}(f, \phi) = \sup \left\{ h_\mu(f) + \int \phi\ d\mu : \mu \text{ is } f\text{-invariant} \right\}$$

where $P_{\text{top}}(f, \phi)$ denotes the topological pressure of f with respect to ϕ and $h_\mu(f)$ denotes the metric entropy. An *equilibrium state* for f with respect to ϕ is an invariant measure that attains the supremum in the right hand side above.

In our setting equilibrium states arise as invariant measures absolutely continuous with respect to an expanding, conformal and non-lacunary Gibbs measure ν. Since we will not use these notions here we shall refer the reader to [VV10] for precise definitions and details. Many important properties arise from the study of transfer operators. Recall that the Ruelle-Perron-Fröbenius transfer operator $\mathcal{L}_{f,\phi}$ associated to $f : M \to M$ and $\phi : M \to \mathbb{R}$ is the linear operator defined on a Banach space $X \subset C^0(M, \mathbb{R})$ of continuous functions $g : M \to \mathbb{R}$ by

$$\mathcal{L}_{f,\phi}(g)(x) = \sum_{f(y)=x} e^{\phi(y)} g(y).$$

Since f is a local homeomorphism it is clear that $\mathcal{L}_{f,\phi}\varphi$ is continuous for every continuous φ and, furthermore, $\mathcal{L}_{f,\phi}$ is indeed a bounded operator relative to the norm of uniform convergence in $C^0(M, \mathbb{R})$ because $\|\mathcal{L}_{f,\phi}\| \leq \deg(f)\, e^{\sup|\phi|}$. Analogously, $\mathcal{L}_{f,\phi}$ preserves the Banach space $C^\alpha(M, \mathbb{R})$, $0 < \alpha < 1$ of Hölder continuous observables. Moreover, it is not hard to check that $\mathcal{L}_{f,\phi}$ is a bounded linear operator in the Banach space $C^r(M, \mathbb{R}) \subset C^0(M, \mathbb{R})$ $(r \geq 1)$ endowed with the norm $\| \cdot \|_r$ whenever f is a C^r-local diffeomorphism and $\phi \in C^r(M, \mathbb{R})$.

We say that the Ruelle-Perron-Frobenius operator $\mathcal{L}_{f,\phi}$ acting on a Banach space X has the *spectral gap property* if there exists a decomposition of its spectrum $\sigma(\mathcal{L}_{f,\phi}) \subset \mathbb{C}$ as follows: $\sigma(\mathcal{L}_{f,\phi}) = \{\lambda_1\} \cup \Sigma_1$ where λ_1 is a leading eigenvalue for $\mathcal{L}_{f,\phi}$ with one-dimensional associated eigenspace and there exists $0 < \lambda_0 < \lambda_1$ such that $\Sigma_1 \subset \{z \in \mathbb{C} : |z| < \lambda_0\}$. When no confusion is possible, for notational simplicity we omit the dependence of the Perron-Frobenius operator on f or ϕ.

We build over the following result which is a consequence of the results in [VV10, CV11]. In what follows, let $\mathcal{F}$ be a family of local homeomorphisms with Lipschitz inverse and let $\mathcal{W}$ be some family of Hölder continuous potentials satisfying (H1), (H2) and (P) with uniform constants. In a similar way given an integer $r \geq 1$, let $\mathcal{F}^r$ be a family of C^r local diffeomorphisms and let $\mathcal{W}^r$ be a family of C^r-potentials satisfying (H1), (H2) and (P') with uniform constants.

Theorem 2.1. *Let $f : M \to M$ be a local homeomorphism with Lipschitz continuous inverse satisfying (H1) and (H2), and let $\phi : M \to \mathbb{R}$ be a Hölder continuous potential such that (P). Then*

1. *there exists a unique equilibrium state μ for f with respect to ϕ, it is expanding, exact and absolutely continuous with respect to some conformal, non-lacunary Gibbs measure ν;*
2. *the Ruelle-Perron-Frobenius has a spectral gap property in the space of Hölder continuous observables and the density $d\mu/d\nu$ is Hölder continuous;*
3. *the topological pressure function $\mathcal{F} \times \mathcal{W} \ni (f, \phi) \to P_{\mathrm{top}}(f, \phi)$ is continuous;*
4. *the invariant density function $\mathcal{F} \times \mathcal{W} \to C^\alpha(M, \mathbb{R})$ given by $(f, \phi) \mapsto \frac{d\mu_{f,\phi}}{d\nu_{f,\phi}}$ is continuous whenever $C^\alpha(M, \mathbb{R})$ is endowed with the C^0 topology.*

If, in addition, the potential $\phi : M \to \mathbb{R}$ is C^r-differentiable and satisfies (P') then

5. *the topological pressure $\mathcal{F}^r \times \mathcal{W}^r \ni (f, \phi) \to P_{\mathrm{top}}(f, \phi)$ and the invariant density function $\mathcal{F}^r \times \mathcal{W}^r \to C^r(M, \mathbb{R})$ given by $(f, \phi) \mapsto \frac{d\mu_{f,\phi}}{d\nu_{f,\phi}}$ vary continuously in the C^r topology;*
6. *the conformal measure function $\mathcal{F}^r \times \mathcal{W}^r \to \mathcal{M}(M)$ given by $(f, \phi) \mapsto \nu_{f,\phi}$ is continuous in the weak* topology. In consequence, the equilibrium measure $\mu_{f,\phi}$ varies continuously in the weak* topology;*

Let us mention that condition (1) above holds more generally for all Hölder continuous potentials such that $\sup\phi - \inf\phi < \log\deg(f) - \log q$ (see [VV10, Theorem A]. Here we build over the aforementioned result by studying finner properties of the transfer operators. Nevertheless, we remark that the these strong statistical stability are not an immediate consequence of the spectral gap property. In fact, the stability of the equilibrium state under deterministic perturbations is more subtle since the Ruelle-Perron-Frobenius operator $\mathcal{L}_{f,\phi}$ acting on the space of Hölder continuous observables is continuous on the potential ϕ but in general it *may not vary continuously* with the underlying dynamics f. We refer the reader to [CV11, Example 4.14] for a detailed discussion. The aforementioned results lead to the natural questions:

Question 1: Is topological pressure differentiable with respect to the dynamical system f and potential ϕ?

Question 2: Are equilibrium states differentiable in the sense that for any given smooth observable $g : M \to \mathbb{R}$ the function

$$(f, \phi) \mapsto \int g \, d\,\mu_{f,\phi}$$

differentiable and can one provide an expression for the derivative?

Question 3: How do Lyapunov exponents and metric entropy associated to equilibrium states $\mu_{f,\phi}$ vary with respect to the dynamics f and potential ϕ?

Question 4: Do large deviation principles holds for Hölder continuous observables ψ, e.g., does there exists a convex rate function $I_{f,\psi}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\phi \Big(x \in M : \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x)) \in [a,b] \Big) = - \inf_{s\in[a,b]} I_{f,\psi}(s)?$$

If so, how does the rate function $I_{f,\psi}$ vary with the dynamical system?

Our purpose in the present paper is to provide answers to these questions in this context of non-uniform expansion.

2.3. **Statement of the main results.** Our first results concerns the differentiability of the spectral radius, associated eigenfunction and eigenmeasures for the Ruelle-Perron-Frobenius and equilibrium states whenever the dynamics f is fixed. Let $C^{r+\alpha}$, with $r \geq 0$ integer and $\alpha \in [0,1)$, denote the space of C^r observables g such that the derivative $D^r g$ is α-Hölder continuous.

Theorem A. *Let $f : M \to M$ be a local homeomorphism with $C^{r+\alpha}$ inverse branches satisfying (H1) and (H2) and let $\mathcal{W} \subset C^{r+\alpha}(M,\mathbb{R})$ be an open subset of Hölder continuous potentials $\phi : M \to \mathbb{R}$ such that (P) holds with uniform constants. Then the following functions are analytic acting on the space $\mathcal{W} \subset C^{r+\alpha}(M,\mathbb{R})$:*

- *The Ruelle-Perron-Frobenius operator function $\mathcal{W} \ni \phi \mapsto \mathcal{L}_\phi \in L(C^{r+\alpha}(M,\mathbb{R}))$;*
- *The spectral radius function $\mathcal{W} \ni \phi \mapsto \lambda_\phi = \exp(P_{\text{top}}(f,\phi))$;*
- *The invariant density function $\mathcal{W} \ni \phi \mapsto h_\phi \in C^{r+\alpha}(M,\mathbb{R})$;*
- *The conformal measure function $\mathcal{W} \ni \phi \mapsto \nu_\phi \in (C^{r+\alpha})^*$. In particular, for any fixed $g \in C^{r+\alpha}(M,\mathbb{R})$ the map $\phi \mapsto \int g \, d\nu_\phi$ is analytic; and*
- *The equilibrium state function $\mathcal{W} \ni \phi \mapsto \mu_\phi = h_\phi \nu_\phi$. In particular, for any fixed $g \in C^{r+\alpha}(M,\mathbb{R})$ the map $\phi \mapsto \int g \, d\mu_\phi$ is analytic.*

Let us mention that some of the previous differentiation formulas hold in general when there is a spectral gap for the transfer operator. We use the differentiability of the Ruelle-Perron-Frobenius operator and the spectral gap property to obtain precise formulas for the previous derivatives.

Theorem B. *Let f and ϕ be as above and assume $r+\alpha>0$. Then for any $n\geq 1$ and $g\in C^{r+\alpha}(M,\mathbb{R})$ be fixed it holds for all $H\in C^{r+\alpha}(M,\mathbb{R})$ that*

- $D_\phi \lambda_{\phi\,|\phi_0}\cdot H=\lambda_{\phi_0}\cdot\int h_{\phi_0}\cdot H\; d\nu_{\phi_0}$;
- $D_\phi h_{\phi\,|\phi_0}\cdot H=h_{\phi_0}\cdot\int\left[(I-\tilde{\mathcal{L}}_{\phi_0\,|_{E_0}})^{-1}(1-h_{\phi_0})\right]\cdot H\; d\nu_{\phi_0}$;
- $D_\phi \int g d\nu_{\phi\,|\phi_0}\cdot H=\int (I-\tilde{\mathcal{L}}_{\phi_0\,|E_0})^{-1}\left(g-\int g d\nu_{\phi_0}\cdot h_{\phi_0}\right)\cdot H d\nu_{\phi_0}$; *and*
-

$$
\begin{aligned}
D_\phi \int g\, d\mu_{\phi\,|\phi_0}\cdot H &= \int (I-\tilde{\mathcal{L}}_{\phi_0\,|E_0})^{-1}\left(g\cdot h_{\phi_0}-\int g\, d\mu_{\phi_0}\cdot h_{\phi_0}\right)\cdot H d\nu_{\phi_0}\\
&\quad+\int g\, d\mu_{\phi_0}\cdot\int\left[(I-\tilde{\mathcal{L}}_{\phi_0\,|E_0})^{-1}(1-h_{\phi_0})\right]\cdot H d\nu_{\phi_0}.
\end{aligned}
$$

Now we shall focus on the C^1-differentiability of the Ruelle-Perron-Frobenius operator, pressure and equilibrium states with respect to the dynamical system f. For short, we say that f is a $C^{r+\alpha}$-local diffeomorphism meaning that f is a C^r-local diffeomorphism and the derivative $D^r f$ is α-Hölder continuous.

Theorem C. *Assume $r\geq 1$ and $\alpha>0$. Let ϕ be a fixed $C^{r+\alpha}$ potential on M and $\mathcal{F}^{r+\alpha}$ be an open subset of $\mathcal{C}^{r+\alpha}$-local diffeomorphisms such that (H1), (H2) and (P') hold with uniform constants. The following properties hold:*

- *For any $g\in C^{r+\alpha}(M,\mathbb{R})$, the map $Diff^{r+\alpha}_{loc}\ni f\mapsto \mathcal{L}^n_f(g)\in C^{r+\alpha}(M,\mathbb{R})$ is differentiable;*
- *The pressure $P_\phi:\mathcal{F}^{r+\alpha}\to\mathbb{R}$ given by $P_\phi(f)=P_{\text{top}}(f,\phi)$ is differentiable;*
- *If $\phi\equiv 0$ then the maximal entropy measure function $\mathcal{F}^{r+\alpha}\ni f\mapsto\mu_f$ is differentiable, that is, the map $\mathcal{F}^{r+\alpha}\ni f\mapsto\int g\; d\mu_f$ is differentiable for any fixed $g\in C^{r+\alpha}(M,\mathbb{R})$.*

In fact, we obtain precise formulas for the previous derivatives as follows.

Theorem D. *Let ϕ be a fixed $C^{r+\alpha}$ potential on M with $\alpha>0$ and let f be a $\mathcal{C}^{r+\alpha}$-local diffeomorphism. Given $g\in C^{r+\alpha}(M,\mathbb{R})$, $H\in C^{r+\alpha}(M,M)$ and $g_1,g_2\in C^{r+\alpha}(M,\mathbb{R})$ we have:*

- *If $\{f_{0,i}: i=1,2,\dots,\deg(f_0)\}$ denotes the set of inverse branches for f_0 then*

$$
\begin{aligned}
\big(D_f\mathcal{L}_{f,\phi}(g)_{|f_0}\cdot H\big)(x) &= \sum_{i=1}^{deg(f_0)} e^{\phi(f_{0,i}(x))}\cdot Dg_{|f_{0,i}(x)}\cdot[(T_{i|f_0}\cdot H)(x)]\\
&\quad+\sum_{i=1}^{deg(f_0)} e^{\phi(f_{0,i}(x))}\cdot g(f_{0,i}(x))\cdot D\phi_{|f_{0,i}(x)}\cdot[(T_{i|f_0}\cdot H)(f_{0,i}(x))];
\end{aligned}
$$

- $D_f\mathcal{L}^n_{f,\phi}(g)_{|f_0}\cdot H=\sum_{i=1}^n\mathcal{L}^{i-1}_{f_0}(D_f\mathcal{L}_{f,\phi}(\mathcal{L}^{n-i}_{f_0}(g))_{|f_0}\cdot H)$;
- $D_f\mathcal{L}^n_{f,\phi}(g_1+tg_2)_{|f_0}\cdot H=D_f\mathcal{L}^n_{f,\phi}(g_1)_{|f_0}\cdot H+tD_f\mathcal{L}^n_{f,\phi}(g_2)_{|f_0}\cdot H$;

- $\|D_f\mathcal{L}_{f,\phi}(g)_{|f_0}\cdot H\|_0 \le \|g\|_1\cdot c_f\cdot\|H\|_1$, *for some* $c_f>0$.

Furthermore, if f satisfies (H1), (H2) and $\phi\equiv 0$ then

- $D_f\mathcal{L}^n_{f,\phi}(1)_{|f_0}\cdot H\equiv 0$;
- $D_f\mu_f(g)_{|f_0}\cdot H=\sum_{i=0}^{\infty}\int D_f\tilde{\mathcal{L}}_f(\tilde{\mathcal{L}}^i_{f_0}(P_0(g)))\cdot H d\mu_{f_0}$.

Using that partial derivatives do exist and are continuous we deduce that the pressure function is globaly differentiable. In particular we obtain the following consequences.

Corollary A. *Assume that $r\ge 1$ and $\alpha>0$. If $\mathcal{F}^{r+\alpha}$ is an open subset of $\mathcal{C}^{r+\alpha}$ local diffeomorphisms such that (H1), (H2) hold and $\mathcal{W}^{1+\alpha}$ is an open set of potentials satisfying (P'), with uniform constants. If $f\mapsto\phi_f\in\mathcal{W}^{1+\alpha}$ is differentiable then the pressure function $f\mapsto P_{\text{top}}(f,\phi_f)$ is differentiable. In particular, if t is small enough the pressure functions $\mathcal{F}^{2+\alpha}\ni f\mapsto P_{\text{top}}(f,-t\log\|Df^{\pm1}\|)$ are differentiable.*

We obtain now consequences for the measure theoretical entropy, extremal Lyapunov exponents and sum of the positive Lyapunov exponents associated to the equilibrium states.

Corollary B. *Assume that $r\ge 1$ and $\alpha>0$. If $\mathcal{F}^{r+\alpha}$ is an open subset of $\mathcal{C}^{r+\alpha}$ local diffeomorphisms such that (H1), (H2) and $\mathcal{W}^{1+\alpha}$ is an open set of potentials satisfying (P'), with uniform constants, then*

$$(f,\phi)\mapsto h_{\mu_{f,\phi}}(f)=P_{\text{top}}(f,\phi)-\int\phi\,d\mu_{f,\phi}$$

and the Lyapunov exponent functions

$$f\mapsto\int\log\|Df(x)\|\,d\mu_{f,\phi}\quad\textit{and}\quad f\mapsto\int\log\|Df(x)^{-1}\|^{-1}\,d\mu_{f,\phi}$$

and

$$f\mapsto\int\log|\det Df(x)|\,d\mu_{f,\phi}$$

are continuous.

As an application of the differentiability of the pressure function we obtain a local large deviations principle. First we recall some notions. Let $\mu_\phi=\mu_{f,\phi}$ be as above. Given an observable $\psi:M\to\mathbb{R}$ and $t\in\mathbb{R}$ the *free energy* $\mathcal{E}_{f,\psi}$ is given by

$$\mathcal{E}_{f,\psi}(t)=\limsup_{n\to\infty}\frac{1}{n}\log\int e^{tS_n\psi}\,d\mu_\phi,$$

where $S_n\psi=\sum_{j=0}^{n-1}\psi\circ f^j$ is the usual Birkhoff sum. In our setting we will prove that the limit above does exist for all Hölder continuous ψ and $|t|\le t_\psi$, for some $t_\psi>0$, and that it is differentiable.

Theorem E. *Let f and ϕ satisfy assumptions (H1), (H2) and (P). Then for any Hölder continuous observable $\psi:M\to\mathbb{R}$ there exists $t_\psi>0$ such that for all $|t|\le t_\psi$ the following limit exists*

$$\mathcal{E}_{f,\psi}(t):=\lim_{n\to\infty}\frac{1}{n}\log\int e^{tS_n\psi}\,d\mu_\phi=P_{\text{top}}(f,\phi+t\psi)-P_{\text{top}}(f,\phi).$$

Moreover, if ψ is cohomologous to a constant then $t \mapsto \mathcal{E}_{f,\psi}(t)$ is linear and, otherwise, $t \mapsto \mathcal{E}_{f,\psi}(t)$ is real analytic and strictly convex. Furthermore, if $f \in Diff^{1+\alpha}_{loc}$ and $\phi, \psi \in C^{1+\alpha}(M, \mathbb{R})$ then for every fixed $t \in \mathbb{R}$ the function $f \mapsto \mathcal{E}_{f,\psi}(t)$ is differentiable and $f \mapsto \mathcal{E}'_{f,\psi}(t)$ is continuous.

So, if ψ is not cohomologous to a constant then the function $[-t_\psi, t_\psi] \ni t \to \mathcal{E}_{f,\psi}(t)$ is strictly convex it is well defined the "local" Legendre transform $I_{f,\psi}$ given by

$$I_{f,\psi}(s) = \sup_{-t_\psi \leq t \leq t_\psi} \big\{ st - \mathcal{E}_{f,\psi}(t) \big\}.$$

We refer the reader to Section 5 for more details. In fact, using differentiability of the pressure function we obtain a level-1 large deviation principle and deduce that the differential variation of the rate function with the dynamical system.

Theorem F. *Let ψ be a Hölder continuous observable not cohomologous to a constant. There exists an interval $J \subset \mathbb{R}$ such that for all $f \in \mathcal{F}$ and $[a, b] \subset J$*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_\phi \left(x \in M : \frac{1}{n} S_n\psi(x) \in [a, b] \right) \leq - \inf_{s\in[a,b]} I_{f,\psi}(s)$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_\phi \left(x \in M : \frac{1}{n} S_n\psi(x) \in (a, b) \right) \geq - \inf_{s\in(a,b)} I_{f,\psi}(s)$$

Moreover, the rate function $(s, f) \mapsto I_{f,\psi}(s)$ is continuous on $J \times \mathcal{F}$.

Let us mention that some upper and lower large deviation bounds for a larger class of transformations were obtained previously in [AP06, Va12] for C^0-observables. The previous provides a local large deviations principle for Hölder observables.

3. Preliminaries

In this section we provide some preparatory results needed for the proof of the main results. Namelly, we recall some properties of the transfer operators.

3.1. **Spectral radius of Ruelle-Perron-Frobenius operators and conformal measures.** Let $\mathcal{L}_\phi : C^0(M, \mathbb{R}) \to C^0(M, \mathbb{R})$ be the Ruelle-Perron-Frobenius transfer operator associated to $f : M \to M$ and $\phi : M \to \mathbb{R}$ previously defined by

$$\mathcal{L}_\phi\varphi(x) = \sum_{f(y)=x} e^{\phi(y)}\varphi(y).$$

for every $\varphi \in C^0(M, \mathbb{R})$. We consider also the dual operator $\mathcal{L}^*_\phi : \mathcal{M}(M) \to \mathcal{M}(M)$ acting on the space $\mathcal{M}(M)$ of Borel measures in M by

$$\int \varphi \, d(\mathcal{L}^*_\phi \eta) = \int (\mathcal{L}_\phi\varphi) \, d\eta$$

for every $\varphi \in C^0(M, \mathbb{R})$. Let $r(\mathcal{L}_\phi)$ be the spectral radius of $\mathcal{L}_\phi$. In our context conformal measures associated to the spectral radius always exist . More precisely,

Proposition 3.1. *Assume that f is topologically exact and (H1), (H2) holds. If ϕ satisfies $\sup\phi - \inf\phi < \log\deg(f) - \log q$ then there exists a conformal measure such that $\mathcal{L}^*_\phi\nu = \lambda\nu$ and $\operatorname{supp}(\nu) = M$, where $\lambda = r(\mathcal{L}_\phi)$. Moreover, ν is a non-lacunary Gibbs measure and $P_{top}(f, \phi) = \log\lambda$.*

Proof. See Theorem B, Theorem 4.1 and Proposition 6.1 in [VV10]. □

3.2. **Spectral gap for the transfer operator in $C^\alpha(M,\mathbb{R})$.** Recall that the Hölder constant of $\varphi \in C^\alpha(M,\mathbb{R})$ is the least constant $C>0$ such that $|\varphi(x)-\varphi(y)| \le C d(x,y)^\alpha$ for all points $x \neq y$ and, for any $\delta>0$, the local Hölder constant $|\varphi|_{\alpha,\delta}$ is the corresponding notion for points x,y such that $d(x,y)<\delta$. If δ is small then there exists a positive integer m such that every (C,α)- Hölder continuous map in balls of radius δ is globally (Cm,α)-Hölder continuous (see [CV11, Lemma 3.5]). This put us in a position to state the precise relation on the constants L, σ, q and ε_ϕ from (H1), (P) and (P'). We assume:

$$e^{\varepsilon_\phi} \cdot \left(\frac{(\deg(f)-q)\sigma^{-\alpha} + qL^\alpha[1+(L-1)^\alpha]}{\deg(f)} \right) + \varepsilon_\phi 2mL^\alpha \operatorname{diam}(M)^\alpha < 1 \tag{3.1}$$

and

$$[1+\varepsilon_\phi] \cdot e^{\varepsilon_\phi} \cdot \left(\frac{(\deg(f)-q)\sigma^{-\alpha} + qL^\alpha[1+(L-1)^\alpha]}{\deg(f)} \right) < 1 \tag{3.2}$$

This choice was enough to obtain the following cone invariance.

Theorem 3.2. *Assume that f satisfies (H1), (H2) and that ϕ satisfies (P). Then there exists $0<\hat\lambda<1$ such that $\mathcal{L}_\phi(\Lambda_{\kappa,\delta}) \subset \Lambda_{\hat\lambda\kappa,\delta}$ for every large positive constant κ, where*

$$\Lambda_{\kappa,\delta} = \left\{ \varphi \in C^0(M,\mathbb{R}) : \varphi > 0 \text{ and } |\varphi|_{\alpha,\delta} \le \kappa \inf \varphi \right\}.$$

is a cone of locally Hölder continuous observables. Moreover, given $0<\hat\lambda<1$, the cone $\Lambda_{\hat\lambda\kappa,\delta}$ has finite $\Lambda_{\kappa,\delta}$-diameter in the projective metric Θ_k. Furthermore, if $\varphi \in \Lambda_{\kappa,\delta}$ satisfies $\int \varphi\, d\nu = 1$ and h denotes the Θ_κ-limit of $\varphi_n = \tilde{\mathcal{L}}_\phi^n(\varphi)$ then, φ_n converges exponentially fast to h in the Hölder norm.

Proof. See Theorem 4.1, Proposition 4.3 and Corollary 4.5 in [CV11]. □

Hence the normalized operator $\tilde{\mathcal{L}}_\phi = \lambda_\phi^{-1}\mathcal{L}_\phi$ has the spectral gap property.

Theorem 3.3. *There exists $0<r_0<1$ such that the operator $\tilde{\mathcal{L}}_\phi$ acting on the space $C^\alpha(M,\mathbb{R})$ admits a decomposition of its spectrum given by $\Sigma = \{1\} \cup \Sigma_0$, where Σ_0 contained in a ball $B(0,r_0)$. In consequence, there exists $C>0$ and $\tau \in (0,1)$ such that $\|\tilde{\mathcal{L}}_\phi^n \varphi - h_\phi \int \varphi\, d\nu_\phi\|_0 \le C\tau^n \|\varphi\|_0$ for all $n \ge 1$ and $\varphi \in C^\alpha(M,\mathbb{R})$, where $h_\phi \in C^\alpha(M,\mathbb{R})$ is the unique fixed point for $\tilde{\mathcal{L}}_\phi$.*

Proof. See Theorem 4.6, Proposition 4.3 and Corollary 4.5 in [CV11]. □

As a consequence of the previous results it follows that the density of the equilibrium state with respect to the corresponding conformal measure vary continuously in the C^0-norm. We recall the precise statement and the proof of the result since some estimates in the proof will be needed later on.

Proposition 3.4. *Let $\mathcal{F}$ be a family of local homeomorphisms and $\mathcal{W}$ be a family of potentials satisfying (H1), (H2) and (P) with uniform constants. Then the topological pressure $\mathcal{F} \times \mathcal{W} \ni (f,\phi) \mapsto \log \lambda_{f,\phi} = P_{top}(f,\phi)$ and the density function*

$$\begin{array}{ccc} \mathcal{F} \times \mathcal{W} & \to & (C^\alpha(M,\mathbb{R}), \|\cdot\|_0) \\ (f,\phi) & \mapsto & \frac{d\mu_{f,\phi}}{d\nu_{f,\phi}} \end{array}$$

are continuous. Moreover, $h_{f,\phi} = \lim \lambda_{f,\phi}^{-n} \mathcal{L}_{f,\phi}^n 1$ and the convergence is uniform in a neighborhood of (f,ϕ).

Proof. Recall that $P_{\rm top}(f,\phi) = \log \lambda_{f,\phi}$ where $\lambda_{f,\phi}$ is the spectral radius of the operator $\mathcal{L}_{f,\phi}$. Moreover, it follows from the proof of Theorem 3.2 that for any $\varphi \in \Lambda_{\kappa,\delta}$ satisfying $\int \varphi \, d\nu = 1$ one has in particular

$$\left\| \lambda_{f,\phi}^{-n} \mathcal{L}_{f,\phi}^{n} \varphi - \frac{d\mu_{f,\phi}}{d\nu_{f,\phi}} \right\|_0 \leq C\tau^n \tag{3.3}$$

for all $n \geq 1$. Notice the previous reasoning applies to $\varphi \equiv 1 \in \Lambda_{\kappa,\delta}$. Moreover, since the spectral gap property estimates depend only on the constants L, σ and $\deg(f)$ it follows that all transfer operators $\mathcal{L}_{\tilde f,\tilde\phi}$ preserve the cone $\Lambda_{\kappa,\delta}$ for all pairs $(\tilde f, \tilde\phi)$ and that the constants R_1 and Δ can be taken uniform in a small neighborhood $\mathcal{U}$ of (f,ϕ). Furthermore, one has that $\int \lambda_{f,\phi}^{-1} \mathcal{L}_{f,\phi} \, d\nu_{f,\phi} = 1$ and so the convergence

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\tilde{\mathcal{L}^n}_{\tilde f,\phi}(1)\|_0 = \lim_{n\to+\infty} \frac{1}{n} \log \left\| {\lambda_{\tilde f,\phi}}^{-n} {\mathcal{L}^n}_{\tilde f,\phi}(1) \right\|_0 = 0$$

given by Theorem 3.2 can be taken uniform in $\mathcal{U}$. This is the key ingredient to obtain the continuity of the topological pressure and density function. Indeed, let $\varepsilon > 0$ be fixed and take $n_0 \in \mathbb{N}$ such that $\left| \frac{1}{n_0} \log \|{\mathcal{L}^{n_0}}_{\tilde f,\phi}(1)\|_0 - \log(\lambda_{\tilde f,\phi}) \right| < \frac{\epsilon}{3}$. for all $\tilde f \in \mathcal{U}$. Moreover, using $P_{\rm top}(f,\phi) = \log \lambda_{f,\phi}$ by triangular inequality we get

$$\begin{aligned}\left| P_{\rm top}(f,\phi) - P_{\rm top}(\tilde f,\phi) \right| \leq & \left| \frac{1}{n_0} \log \|{\mathcal{L}^{n_0}}_{\tilde f,\phi}(1)\|_0 - \log(\lambda_{\tilde f,\phi}) \right| \\ & + \left| \frac{1}{n_0} \log \|{\mathcal{L}^{n_0}}_{f,\phi}(1)\|_0 - \log(\lambda_{f,\phi}) \right| \\ & + \left| \frac{1}{n_0} \log \|{\mathcal{L}^{n_0}}_{f,\phi}(1)\|_0 - \frac{1}{n_0} \log \|{\mathcal{L}^{n_0}}_{\tilde f,\phi}(1)\|_0 \right|. \end{aligned}$$

Now, it is not hard to check that, for n_0 fixed, the function $\mathcal{U} \to C^0(M,\mathbb{R})$

$$\tilde f \mapsto \mathcal{L}_{\tilde f,\phi}^{n_0} 1 = \sum_{\tilde f^{n_0}(y)=x} e^{S_{n_0}\phi(y)}$$

is continuous. Consequently, there exists a neighborhood $\mathcal{V} \subset \mathcal{U}$ of f such that $|\frac{1}{n_0} \log \|{\mathcal{L}^{n_0}}_{f,\phi}(1)\|_0 - \frac{1}{n_0} \log \|{\mathcal{L}^{n_0}}_{\tilde f,\phi}(1)\|_0| < \varepsilon/3$ for every $\tilde f \in \mathcal{V}$. Altogether this proves that $|P_{\rm top}(f,\phi) - P_{\rm top}(\tilde f,\phi)\Big| < \varepsilon$ for all $\tilde f \in \mathcal{V}$. Since ε was chosen arbitrary we obtain that both the leading eigenvalue and topological pressure functions vary continuously with the dynamics f. Finally, by equation (3.3) above applied to $\varphi \equiv 1$ and triangular inequality we obtain that

$$\left\| \frac{d\mu_{\tilde f,\phi}}{d\nu_{\tilde f,\phi}} - \frac{d\mu_{f,\phi}}{d\nu_{f,\phi}} \right\|_0 \leq 2C\tau^n + \left\| \lambda_{\tilde f,\phi}^{-n} \mathcal{L}_{\tilde f,\phi}^{n} 1 - \lambda_{f,\phi}^{-n} \mathcal{L}_{f,\phi}^{n} 1 \right\|_0$$

for all n. Hence, proceeding as before one can make the right hand side above as close to zero as desired provided that $\tilde f$ is sufficiently close to f. This proves the continuity of the density function and finishes the proof of the proposition. □

3.3. **Spectral gap for the transfer operator in $C^r(M,\mathbb{R})$.** In this subsection we recall the analogous results for the action of the transfer operator in the space of smooth observables. In particular we have the corresponding spectral gap property for the action of Ruelle-Perron-Frobenius transfer operators in the space of smooth observables whose proof can be found in [CV11, Section 5].

Theorem 3.5. *There exists $0 < r_0 < 1$ such that the operator $\tilde{\mathcal{L}}_\phi$ acting on the space $C^r(M,\mathbb{R})$ $(r \geq 1)$ admits a decomposition of its spectrum given by $\Sigma = \{1\} \cup \Sigma_0$, where Σ_0 contained in a ball $B(0, r_0)$. In consequence, there exists $C > 0$ and $\tau \in (0,1)$ such that $\|\tilde{\mathcal{L}}_\phi^n \varphi - h_\phi \int \varphi \, d\nu_\phi\|_r \leq C\tau^n \|\varphi\|_r$ for all $n \geq 1$ and $\varphi \in C^r(M,\mathbb{R})$, where $h_\phi \in C^r(M,\mathbb{R})$ is the unique fixed point for $\tilde{\mathcal{L}}_\phi$.*

Let us mention also that by Proposition 5.4 in [CV11] one has that

$$P_{\text{top}}(f,\phi) = \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}^n 1\|_r$$

and that the limit can be taken uniform in a C^r neighborhood of (f,ϕ). This is the key fact that will be used later on to prove the differentiability of the topological pressure.

4. Differentiability results

4.1. **Differentiation with respect to the potential.** First we fix f and will focus on the differentiability questions with respect to the potential ϕ. Let $\mathcal{W}$ be an open set of potentials in $C^\alpha(M,\mathbb{R})$, $\alpha > 0$, satisfying the condition (P), endowed with the C^α- topology. For notational simplicity, when no confusion is possible, we write simply $\mathcal{L}_\phi$, λ_ϕ and h_ϕ omitting the dependence on f. The following results hold generally when the transfer operator has the spectral gap property.

Proposition 4.1. *Assume that $r + \alpha > 0$. The map $C^{r+\alpha} \ni \phi \mapsto \mathcal{L}_\phi^n \in \mathcal{L}(C^{r+\alpha})$ is analytic, hence C^∞. Moreover, for every vectors $g, H \in C^{r+\alpha}$, $\|g\|_{r+\alpha} = 1$ and for every $n \geq 1$, the first derivative acting in H is given by*

$$(D_\phi \mathcal{L}_\phi^n(g))_{|\phi_0}(H) = \sum_{i=1}^{n} \mathcal{L}_{\phi_0}^i (H \cdot \mathcal{L}_{\phi_0}^{n-i}(g)).$$

Proof. Note that

$$\mathcal{L}_{\phi+H}(g) = \mathcal{L}_\phi(e^H g) = \sum_{i=0}^{\infty} \mathcal{L}_\phi\left(\frac{1}{i!} H^i g\right) = \mathcal{L}_\phi(g) + \sum_{i=1}^{\infty} \frac{1}{i!} \mathcal{L}_\phi\left(H^i g\right).$$

Let us denote by $\mathcal{L}_s^i(C^{r+\alpha}, C^{r+\alpha})$ the space of symmetric i-linear maps with domain in $[C^{r+\alpha}]^i$ into $C^{r+\alpha}$. Note also that the maps

$$C^{r+\alpha} \ni \phi \mapsto \Big(H \mapsto \mathcal{L}_\phi(H^i \cdot)\Big) \in \mathcal{L}_s^i(C^{r+\alpha})$$

are continuous for every $i \in \mathbb{N}$, and that the product between functions is also continuous in $C^{r+\alpha}$. Therefore for $k \in \mathbb{N}$, it follows that

$$\sup_{\|g\|_{r+\alpha}=1} \frac{\|\mathcal{L}_{\phi+H}(g) - \mathcal{L}_\phi(g) - \sum_{i=1}^{k} \frac{1}{i!} \mathcal{L}_\phi\left(H^i g\right)\|_{r+\alpha}}{\|g\|_{r+\alpha} \|H\|_{r+\alpha}^k} \leq \sum_{i=k+1}^{\infty} \sup_{\|g\|_{r+\alpha}=1} \frac{\|\frac{1}{i!} \mathcal{L}_\phi\left(H^i g\right)\|_{r+\alpha}}{\|g\|_{r+\alpha} \|H\|_{r+\alpha}^k}$$

which converges to zero as H tends to zero. By Theorem 1.4 in [Fr79], this implies that $\phi \mapsto \mathcal{L}_\phi$ is C^k, for any $k \in \mathbb{N}$, and its k-th derivative applied in H is $\mathcal{L}_\phi\left(H^i \cdot\right)$. Note that this also implies that $\phi \mapsto \mathcal{L}_\phi$ is analytic. By applying the chain rule to the composition $\phi \mapsto \mathcal{L}_\phi^n(g)$ we finish the proof of the proposition. $\square$

Remark 4.2. Our previous argument implies in particular that for any given $g \in C^\alpha(M,\mathbb{R})$ the map $t \mapsto \mathcal{L}_{t\phi}$ is real analytic since our previous arguments lead to

$$\mathcal{L}_{t\phi}(g) = \sum_{i=0}^{\infty} \mathcal{L}_\phi\left(\frac{1}{i!}[(1-t)\phi]^i g\right) = \mathcal{L}_\phi(g) + \mathcal{L}_\phi((1-t)\phi g) + \sum_{i=2}^{\infty} \mathcal{L}_\phi\left(\frac{1}{i!}[(1-t)\phi]^i g\right)$$

and the series is convergent.

Let us mention that [VV10] proved that the sequence $\frac{1}{n}\sum_{j=0}^{n-1} f_*^j \nu$ converges to the unique equilibrium state μ. Here we deduce much stronger properties fundamental to the proof that the spectral radius of the Ruelle-Perron-Frobenius operator varies differentiably with respect to the potential ϕ. We show that $(\tilde{\mathcal{L}}^n)^*\xi$ converge exponentially fast to ν for any probability measure $\xi \in \mathcal{M}(M)$. More precisely,

Proposition 4.3. *There exists $C > 0$ and $\tau \in (0,1)$ such that for every $\varphi \in C^\alpha(M,\mathbb{R})$ and every probability measure $\xi \in \mathcal{M}(M)$ it holds that*

$$\left| \int \varphi \, d(\tilde{\mathcal{L}}^n)^*\xi - \int h_\phi \, d\xi \int \varphi \, d\nu_\phi \right| \le C\tau^n \|\varphi\|_0.$$

Proof. The proof is a simple consequence of the spectral gap property. In fact,

$$\begin{aligned} \left| \int \varphi \, d(\tilde{\mathcal{L}}^n)^*\xi - \int h_\phi d\xi \int \varphi d\nu_\phi \right| &\le \int \left| \tilde{\mathcal{L}}^n(\varphi) - h_\phi \int \varphi d\nu_\phi \right| d\xi \\ &\le \left\| \tilde{\mathcal{L}}^n(\varphi) - h_\phi \int \varphi d\nu_\phi \right\|_0 \le C\tau^n \|\varphi\|_0, \end{aligned}$$

where C and τ are given by Theorem 3.3. This proves our proposition. □

Proposition 4.4. *The spectral radius map $\mathcal{W} \ni \phi \mapsto \lambda_\phi$ is analytic. Furthermore, given $\phi_0 \in \mathcal{W}$ and $H \in C^\alpha(M,\mathbb{R})$ we have:*

$$D_\phi \lambda_{\phi \; |\phi_0} \cdot H = \lambda_{\phi_0} \cdot \int h_{\phi_0} \cdot H \; d\nu_{\phi_0}.$$

Proof. Note that it is immediate that $\phi \mapsto \lambda_\phi$ is analytic, since $\phi \mapsto \mathcal{L}_\phi$ is analytic (in the norm operator topology), and since the spectral radius of $\mathcal{L}_\phi$ coincides with an isolated eigenvalue of $\mathcal{L}_\phi$ with multiplicity one. Let us calculate explicitly the derivative of $\phi \mapsto \lambda_\phi$.

Let $\phi_0 \in \mathcal{W}$ be fixed. It follows from the C^0-statistical stability statement in Proposition 3.4 that $\lambda_\phi^{-n}\mathcal{L}_\phi^n(1) \to h_\phi$ and that the limit is uniform in a small neighborhood W of ϕ_0. Moreover, since $\tilde{\mathcal{L}}_\phi^n(1)(x) \le K$ for some constant K that can be taken uniform in W it follows that h_ϕ can be taken uniformly bounded from above for all $\phi \in W$. Since the sequence $\tilde{\mathcal{L}}_\phi^n(1)$ is Cauchy in the projective metric it also follows that h_ϕ can be taken uniformly bounded away from zero for all $\phi \in W$. In consequence, $\lim_{n\to\infty} \frac{1}{n} \log \int \mathcal{L}_\phi^n 1 \; d\nu_{\phi_0} = \log \lambda_\phi$ uniformly with respect to $\phi \in W$. Hence, we consider the family of functionals $F_n : W \to \mathbb{R}$ given by

$$F_n(\phi) = \frac{1}{n} \log \int \mathcal{L}_\phi^n 1 \, d\nu_{\phi_0},$$

which are well defined and converge to the constant $\log \lambda_\phi$, and prove that the derivatives of F_n converge uniformly as n tends to infinity. One can write

$$DF_n(\phi)\cdot H = \frac{\int D_\phi \mathcal{L}^n_\phi(1)_{|\phi} \cdot H d\nu_{\phi_0}}{n \cdot \int \mathcal{L}^n_\phi(1) d\nu_{\phi_0}} = \frac{\int \sum_{i=1}^n \mathcal{L}^i_\phi(\mathcal{L}^{n-i}_\phi(1)\cdot H)\, d\nu_\phi}{n\cdot \int \mathcal{L}^n_\phi(1) d\nu_{\phi_0}} = \frac{A_n(\hat{\phi})\cdot H}{\int \tilde{\mathcal{L}}^n_{\hat{\phi}}(1) d\nu_{\phi_0}}, \tag{4.1}$$

where A_n an averaged sum using the normalized operators $\mathcal{L}_\phi = \lambda_\phi^{-1}\mathcal{L}_\phi$ defined by

$$A_n(\phi)\cdot H = \frac{1}{n}\int \sum_{i=1}^n \tilde{\mathcal{L}}^i_\phi(\tilde{\mathcal{L}}^{n-i}_\phi(1)\cdot H)\, d\nu_{\phi_0}.$$

Taking into account Proposition 4.3 it follows that

$$\begin{aligned}
&|A_n(\phi)\cdot H - \frac{1}{n}\sum_{i=0}^{n-1}\int \tilde{\mathcal{L}}^i_\phi(1)\cdot H d\nu_\phi \cdot \int h_\phi d\nu_{\phi_0}| \\
&\qquad \le \frac{1}{n}\sum_{i=1}^n \left| \int \tilde{\mathcal{L}}^{n-i}_\phi(1)\cdot H\; d(\tilde{\mathcal{L}}^{*i}_\phi \nu_{\phi_0}) - \int \tilde{\mathcal{L}}^{n-i}_\phi(1)\cdot H\; d\nu_\phi \cdot \int h_\phi d\nu_{\phi_0}\right| \\
&\qquad \le \frac{1}{n}\sum_{i=1}^n k\tau^i \cdot \|\tilde{\mathcal{L}}^{n-i}_\phi(1)\cdot H\|_0 \le \frac{1}{n}\sum_{i=1}^n k\tau^i\cdot(k\tau^{n-i} + \|h_\phi\|_0)\cdot\|H\|_0,
\end{aligned}$$

which is uniformly convergent to zero with respect to ϕ and unitary vectors $H \in C^\alpha(M,\mathbb{R})$. Furthermore,

$$\frac{1}{n}\sum_{i=0}^{n-1}\int \tilde{\mathcal{L}}^i_\phi(1)\cdot H d\nu_\phi \cdot \int h_\phi d\nu_{\phi_0} \xrightarrow[n\to\infty]{} \int h_\phi \cdot H d\nu_\phi \cdot \int h_\phi d\nu_{\phi_0}$$

and this convergence is uniform with respect to ϕ e H. Since $\int \tilde{\mathcal{L}}^n_\phi 1\, d\nu_{\phi_0}$ converges to $\int h_\phi d\nu_{\phi_0}$ uniformly with respect to ϕ, we obtain that

$$DF_n(\phi)\cdot H = \frac{A_n(\phi)\cdot H}{\int \tilde{\mathcal{L}}^n_\phi(1) d\nu_{\phi_0}} \to \int h_\phi \cdot H\; d\nu_\phi,$$

where the convergence is uniform with respect to ϕ and $H \in C^\alpha(M,\mathbb{R})$ satisfying $\|H\|_\alpha = 1$. Now, just observe that $e^{F_n(\phi)}$ is differentiable and uniformly convergent to $\lambda_{f,\phi}$. Thus, as a consequence of the chain rule it follows that

$$D_\phi \lambda_{\phi\ |\phi_0} \cdot H = \lambda_{\phi_0} \cdot \int h_{\phi_0}\cdot H\; d\nu_{\phi_0}.$$

This finishes the proof of the proposition. $\square$

Since in our context it follows that $P_{\text{top}}(f,\phi) = \log \lambda_{f,\phi}$ and the arguments in the later proof lead to the following immediate consequence:

Corollary 4.5. *The map $\mathcal{W} \ni \phi \mapsto P_{top}(f,\phi)$ is differentiable. Furthermore, given $\phi_0 \in \mathcal{W}$ and $H \in C^\alpha(M,\mathbb{R})$ we have:*

$$D_\phi P_{top}(f,\phi)_{\ |\phi_0} \cdot H = \int h_{\phi_0}\cdot H\; d\nu_{\phi_0} = \int H\; d\mu_{\phi_0}.$$

From Proposition 3.4 the invariant density is Hölder continuous function and varies continuously in the C^0-topology. Here we show that it varies differentiably with respect to the potential.

Proposition 4.6. *The map $\mathcal{W} \ni \phi \mapsto h_\phi \in C^0(M, \mathbb{R})$ is differentiable. Furthermore, given $\phi_0 \in \mathcal{W}$ and $H \in C^\alpha(M, \mathbb{R})$ we have:*

$$D_\phi h_{\phi\;|\phi_0} \cdot H = h_{\phi_0} \cdot \int \big[(I - \tilde{\mathcal{L}}_{\phi_0\;|E_0})^{-1}(1 - h_{\phi_0})\big] \cdot H \; d\nu_{\phi_0}.$$

Proof. Let $\phi_0 \in \mathcal{W}$ be fixed. By the C^0-statistical stability Theorem we have that $\lambda_{f,\phi}^{-n} \mathcal{L}_{f,\phi}^n 1$ converges uniformly to $h_{f,\phi}$ with respect to respect to ϕ in some suficiently small neighborhood W of ϕ_0. Hence, consider as before a a familly of functionals $F_n : W \to \mathbb{R}$ given by $F_n(\phi) = \lambda_{f,\phi}^{-n}\, \mathcal{L}_{f,\phi}^n 1$. We claim that the derivatives of F_n are uniformly convergent. Indeed, taking into account Proposition 4.4 for the derivative of the spectral radius λ_ϕ one can write

$$\begin{aligned} DF_n(\phi) \cdot H &= \lambda_\phi^{-n} D_\phi \mathcal{L}_\phi^n 1_{|\phi} \cdot H - n \lambda_\phi^{-(n+1)} \mathcal{L}_\phi^n 1 \cdot D_\phi \lambda_{\phi|\phi} \cdot H \\ &= \left[\sum_{i=1}^{n} \tilde{\mathcal{L}}_\phi^i (\tilde{\mathcal{L}}_\phi^{n-i}(1) \cdot H) - n \tilde{\mathcal{L}}_\phi^n(1) \cdot \int h_\phi \cdot H \; d\nu_\phi \right]. \end{aligned} \tag{4.2}$$

We now prove that the later expression is uniformly convergent (with respect to ϕ and $\{H \in C^\alpha(M, \mathbb{R}) : \|H\|_\alpha \leq 1\}$) to the series $h_\phi \cdot \sum_{i=0}^{+\infty} \int (\tilde{\mathcal{L}}_\phi^i(1) \cdot H - h_\phi \cdot H) \; d\nu_\phi$. On the one hand,

$$\begin{aligned} h_\phi \cdot \sum_{i=0}^{n-1} \int (\tilde{\mathcal{L}}_\phi^i(1) \cdot H - h_\phi \cdot H) \; d\nu_\phi &= h_\phi \cdot \int \sum_{i=0}^{n-1} \tilde{\mathcal{L}}_\phi^i (1 - h_\phi) \cdot H \; d\nu_\phi \\ &\to h_\phi \cdot \int \big[(I - \tilde{\mathcal{L}}_{\phi\;|E_0})^{-1}(1 - h_\phi)\big] \cdot H \; d\nu_\phi \end{aligned}$$

as n tends to infinite. On the other hand, using (4.2) one can write

$$\begin{aligned} & DF_n(\phi) \cdot H - h_\phi \cdot \sum_{i=0}^{n-1} \int [\tilde{\mathcal{L}}_\phi^i(1) \cdot H - \int h_\phi \cdot H] \; d\nu_\phi \\ &= \sum_{i=1}^{n} \tilde{\mathcal{L}}_\phi^i \big[\tilde{\mathcal{L}}_\phi^{n-i}(1) H - \tilde{\mathcal{L}}_\phi^{n-i}(1) \int h_\phi H \; d\nu_\phi - h_\phi \int (\tilde{\mathcal{L}}_\phi^{n-i}(1) H - h_\phi H) \; d\nu_\phi \big] \end{aligned}$$

and is equal to a sum $\sum_{i=1}^n \tilde{\mathcal{L}}_\phi^i(\xi_n)$ where the functions ξ_n have zero mean average with respect to the conformal measure ν_ϕ. Therefore it follows from Theorem 3.3 that there exists $C > 0$ and $0 < \tau < 1$ such that

$$\begin{aligned} & \|DF_n(\phi) \cdot H - h_\phi \cdot \sum_{i=0}^{n-1} \int [\tilde{\mathcal{L}}_\phi^i(1) \cdot H - \int h_\phi \cdot H] \; d\nu_\phi\|_0 \\ &\quad \leq \sum_{i=1}^{n} C\tau^i \|\tilde{\mathcal{L}}_\phi^{n-i}(1) H - \tilde{\mathcal{L}}_\phi^{n-i}(1) \int h_\phi H \; d\nu_\phi - h_\phi \int (\tilde{\mathcal{L}}_\phi^{n-i}(1) H - h_\phi H) \; d\nu_\phi\|_0 \\ &\quad \leq \sum_{i=1}^{n} C\tau^i \|\tilde{\mathcal{L}}_\phi^{n-i}(H \circ f^{n-i}) - h_\phi \cdot \int \tilde{\mathcal{L}}_\phi^{n-i}(H \circ f^{n-i}) \; d\nu_\phi\| \\ &\quad + \sum_{i=1}^{n} C\tau^i \|\tilde{\mathcal{L}}_\phi^{n-i}(1) \cdot \int h_\phi \cdot H \; d\nu_\phi - h_\phi \cdot \int h_\phi \cdot H \; d\nu_\phi\|_0 \\ &\quad \leq \sum_{i=1}^{n} C^2 \tau^n \cdot \|H\|_0 \cdot (1 + \|h_\phi\|_0)^2, \end{aligned}$$

which is convergent to zero. Since all convergences above are uniform with respect to ϕ and $H \in C^\alpha(M,\mathbb{R})$ with $\|H\|_\alpha \le 1$ the previous estimates prove our claim. Now we can finish the proof of the proposition by estimating

$$DF_n(\phi)\cdot H \to \int \left[(I-\tilde{\mathcal{L}}_{\phi\,|E_0})^{-1}(1-h_\phi)\right]\cdot H\; d\nu_\phi \cdot h_\phi,$$

uniformly with respect to ϕ and $H \in C^\alpha(M,\mathbb{R})$, $\|H\|_\alpha = 1$. Thus we deduce that

$$D_\phi h_{\phi\,|\phi_0}\cdot H = h_{\phi_0}\cdot \int \left[(I-\tilde{\mathcal{L}}_{\phi_0\,|E_0})^{-1}(1-h_{\phi_0})\right]\cdot H\; d\nu_{\phi_0},$$

which finishes the proof of the proposition. □

Now we use the previous information to deduce that the conformal measures ν_ϕ are differentiable. The precise statement is as follows:

Proposition 4.7. *Let $g \in C^\alpha(M,\mathbb{R})$ be fixed. The map $\mathcal{W} \ni \phi \mapsto \int g\; d\nu_\phi$ is differentiable. Furthermore, given $\phi_0 \in \mathcal{W}$ and $H \in C^\alpha(M,\mathbb{R})$ we have:*

$$D_\phi \int g\; d\nu_{\phi\,|\phi_0}\cdot H = \int (I-\tilde{\mathcal{L}}_{\phi_0\,|E_0})^{-1}\left(g-\int g\; d\nu_{\phi_0}\cdot h_{\phi_0}\right)\cdot H d\nu_{\phi_0}.$$

Proof. Fix any $g \in C^\alpha(M,\mathbb{R})$ and $\phi_0 \in \mathcal{W}$. Since we deal with differentiability conditions it is enough to consider the sequence of functionals F_n defined in a small neighborhood W of ϕ_0 by

$$F_n(\phi) := \frac{\int \mathcal{L}_\phi^n g\; d\nu_{\phi_0}}{\int h_\phi\; d\nu_{\phi_0}}.$$

If the neighorhood W is small enough then we have that $F_n(\phi)$ converges uniformly to $\int g\; d\nu_\phi$. Moreover, for any $H \in C^\alpha(M,\mathbb{R})$, it is not hard to check that

$$DF_n(\phi)\cdot H = \frac{A_n(\phi)}{\int h_\phi d\nu_{\phi_0}} - B_n(\phi)$$

where the formal series A_n and B_n are defined as

$$A_n(\phi)\cdot H := \int \Big[\sum_{i=1}^n \tilde{\mathcal{L}}_\phi^i(\tilde{\mathcal{L}}_\phi^{n-i}(g)\cdot H) - n\int h_\phi\cdot H d\nu_\phi\cdot \tilde{\mathcal{L}}_\phi^n g\Big] d\nu_{\phi_0}.$$

and

$$B_n(\phi)\cdot H := \frac{\int (D_\phi h_{\phi\,|\phi}\cdot H)d\nu_{\phi_0}\cdot \int \tilde{\mathcal{L}}_\phi^n(g)d\nu_{\phi_0}}{\int h_\phi d\nu_{\phi_0}}$$

respectively. We proceed to establish the uniform convergence of these formal series. First we note that as before one can write

$$\begin{aligned} A_n(\phi)\cdot H - \sum_{i=0}^{n-1}\int\left(\tilde{\mathcal{L}}_\phi^i(g)\cdot H - \int g d\nu_\phi\cdot h_\phi\cdot H\right)d\nu_\phi\cdot\int h_\phi d\nu_{\phi_0} \\ = \sum_{i=1}^n \int \tilde{\mathcal{L}}_\phi^i\Bigg(\tilde{\mathcal{L}}_\phi^{n-i}(g)H - \tilde{\mathcal{L}}_\phi^{n-i}(g)\int h_\phi H\; d\nu_\phi \\ - \int(\tilde{\mathcal{L}}_\phi^{n-i}(g)H - \int g d\nu_\phi h_\phi H)d\nu_\phi h_\phi\Bigg)\; d\nu_{\phi_0}. \end{aligned}$$

Thus, proceeding as in the proof of Proposition 4.6 we get that

$$\left\| A_n(\phi) \cdot H - \sum_{i=0}^{n-1} \int \left(\tilde{\mathcal{L}}_\phi^i(g) \cdot H - \int g d\nu_\phi \cdot h_\phi \cdot H \right) d\nu_\phi \cdot \int h_\phi d\nu_{\phi_0} \right\|_0$$

is bounded from above by

$$\begin{aligned}
&\sum_{i=1}^{n} k\tau^i \left\| \tilde{\mathcal{L}}_\phi^{n-i}(g)H - \tilde{\mathcal{L}}_\phi^{n-i}(g) \int h_\phi H \, d\nu_\phi - \int (\tilde{\mathcal{L}}_\phi^{n-i}(g)H - \int g d\nu_\phi h_\phi H) d\nu_\phi h_\phi \right\|_0 \\
&\qquad \le \sum_{i=1}^{n} k\tau^i \| \tilde{\mathcal{L}}_\phi^{n-i}(gH \circ f^{n-i}) - h_\phi \int \tilde{\mathcal{L}}_\phi^{n-i}(gH \circ f^{n-i}) d\nu_\phi \| \\
&\qquad + \sum_{i=1}^{n} k\tau^i \| \tilde{\mathcal{L}}_\phi^{n-i}(g) \int h_\phi H \, d\nu_\phi \| - \int g d\nu_\phi \int h_\phi H \, d\nu_\phi h_\phi \|_0 \\
&\qquad \le \sum_{i=1}^{n} k^2 \tau^n \|g\|_0 \cdot \|H\|_0 \cdot (1 + \|h_\phi\|_0)^2,
\end{aligned}$$

which is uniformly convergent to zero with respect to ϕ and unitary vectors $H \in C^\alpha(M, \mathbb{R})$. Thus we get that $A_n(\phi) \cdot H$ is uniformly convergent to

$$\sum_{i=0}^{+\infty} \int \left(\tilde{\mathcal{L}}_\phi^i(g) \cdot H - \int g d\nu_\phi \cdot h_\phi \cdot H \right) d\nu_\phi \cdot \int h_\phi d\nu_{\phi_0}. \tag{4.3}$$

As for B_n it is not hard to check that $B_n(\phi) \cdot H$ is uniformly convergent to

$$\frac{\int (D_\phi h_{\phi \,|\phi} \cdot H) d\nu_{\phi_0} \cdot \int g d\nu_\phi}{\int h_\phi d\nu_{\phi_0}} = \sum_{i=0}^{+\infty} \int (\tilde{\mathcal{L}}_\phi^i(1) \cdot H - h_\phi \cdot H) \, d\nu_\phi \cdot \int g d\nu_\phi \tag{4.4}$$

uniformly with respect to ϕ and $H \in C^\alpha(M, \mathbb{R}) : \|H\|_\alpha = 1$. Therefore the result follows by combining equality (4.1) above with the limit series (4.3) and (4.4) since $DF_n(\phi) \cdot H$ is uniformly convergent with respect to ϕ and $H \in C^\alpha(M, \mathbb{R}), \|H\|_\alpha = 1$ to the expression

$$\sum_{i=0}^{+\infty} \int \left[\tilde{\mathcal{L}}_\phi^i(g) - \tilde{\mathcal{L}}_\phi^i(\int g d\nu_\phi \cdot h_\phi) \right] \cdot H d\nu_\phi = \int \left[(I - \tilde{\mathcal{L}}_{\phi \,|E_0})(g - \int g d\nu_\phi \cdot h_\phi) \right] \cdot H d\nu_\phi.$$

Thus we get as claimed

$$D_\phi \int g d\nu_{\phi \,|\phi_0} \cdot H = \int \left[(I - \tilde{\mathcal{L}}_{\phi_0 \,|E_0})^{-1} (g - \int g d\nu_{\phi_0} \cdot h_{\phi_0}) \right] \cdot H d\nu_{\phi_0}.$$

□

We will now deduce the differentiability of the equilibrium states μ_ϕ with respect to the potential ϕ. In fact, using that $\mu_\phi = h_\phi \ \nu_\phi$ the following consequence is immediate from our previous two differentiability results.

Corollary 4.8. *Given $g \in C^\alpha(M, \mathbb{R})$, the map $\mathcal{W} \ni \phi \mapsto \int g \, d\mu_\phi$ is differentiable. Furthermore, given $\phi_0 \in \mathcal{W}$ and $H \in C^\alpha(M, \mathbb{R})$ we have*

$$\begin{aligned}
D_\phi \int g \, d\mu_{\phi \,|\phi_0} \cdot H &= \int (I - \tilde{\mathcal{L}}_{\phi_0 \,|E_0})^{-1} \left(g \cdot h_{\phi_0} - \int g \, d\mu_{\phi_0} \cdot h_{\phi_0} \right) \cdot H d\nu_{\phi_0} \\
&\quad + \int g \, d\mu_{\phi_0} \cdot \int \left[(I - \tilde{\mathcal{L}}_{\phi_0 \,|E_0})^{-1} (1 - h_{\phi_0}) \right] \cdot H d\nu_{\phi_0}.
\end{aligned}$$

4.2. **Differentiability of the topological pressure with respect to the dynamics.** In this subsection we prove the differentiability of the topological pressure using the differentiability of inverse branches for the dynamics. More precisely,

Lemma 4.9. *(Local Differentiability of inverse branches) Let $r \geq 1$ and $f : M \to M$ be a C^r-local diffeomorphism on a compact connected manifold M. Let $B = B(x,\delta) \subset M$ some ball such that the inverse branches $f_1, \ldots, f_s : B \to M$ are well defined diffeomorphism onto their images. Then $f \mapsto (f_1, \ldots, f_s)$ is a C^{r-1} map.*

Proof. Let $F : C^r(M,M) \times [C^{r-1}(B,M)]^s \to [C^{r-1}(B;M)]^s$ given by

$$F(h, \underbrace{h_1, \ldots, h_s}_{:=\underline{h}}) = (h \circ h_1, \ldots, h \circ h_s).$$

Note that F is C^r. In fact, on one hand, $\partial_h F$ is C^∞ (in suitable charts, we see it as a continuous linear map in h). On the other hand, by taking an increment $H = (H_1, \ldots, H_s) \in [C^{r-1}(B;M)]^s$, we obtain that $\partial_{h_j} F \cdot H_j = h' \circ h_j \cdot H_j$, which is clearly a C^{r-1} map.

Note that $F(f, f_1, \ldots, f_s) = (id, \ldots, id)$. For the point $(f, f_1, \ldots, f_s)$, we have that $\partial_{\underline{h}} F(f, f_1, \ldots, f_s) \cdot H = (f' \circ f_1 \cdot H_1, \ldots, f' \circ f_s \cdot H_s)$, is an isomorphism, since f is a local diffeomorphism and so $[f' \circ g_1(x)]$ is invertible, for any $x \in M$. Therefore, by Implicit Function Theorem, we obtain that the map $G : C^r(M,M) \to [C^{r-1}(B,M)]^s$ given by $f \mapsto (f_1, \ldots, f_s)$ is a C^{r-1} map, and its derivative applied to an increment $h \in C^r(M,M)$ is

$$(DG \cdot h)(x) = (-f_1'(x) \cdot h \circ f_1(x), \ldots, -f_s'(x) \cdot h \circ f_s(x))$$

This finishes the proof of the lemma. $\square$

Using the differentiability of the inverse branches we can now establish the differentiability of the transfer operator with respect to the underlying dynamics.

Lemma 4.10. *(Differentiability of transfer operator) Let $r \geq 1$, $f : M \to M$ be a C^r-local diffeomorphism on a compact connected manifold M and $\phi \in C^r(M,\mathbb{R})$ be any fixed potential. For any $g \in C^r(M,\mathbb{R})$ the map*

$$\begin{array}{ccc} C^r(M,\mathbb{R}) & \to & C^{r-1}(M,\mathbb{R}) \\ f & \mapsto & \mathcal{L}_{f,\phi}(g) \end{array}$$

is differentiable.

Proof. Let $\{\varphi_j, j = 1, \ldots, l\}$ be a C^∞ partition of unity associated to some finite covering $B_1, \ldots B_l$ of M by balls with radius smaller or equal to $\delta > 0$ and define the auxiliar operators $\mathcal{L}_j = \mathcal{L}_{j,f,\phi} := \mathcal{L}_{f,\phi} \cdot \varphi_j$. In particular it holds that $\mathcal{L}_{f,\phi} = \sum_{j=1}^l \mathcal{L}_j$. Therefore, all we need to prove is that any auxiliar operator $\mathcal{L}_j$ is pointwise differentiable. Let us write $B = B_j$ and recall that φ_j vanishes outside B. We also write $f_1, \ldots, f_s$ for the inverse branches of f in B, and let us write $T_i = \partial_f f_i$, for $i = 1, \ldots, s$. Therefore, we have:

$$\mathcal{L}_j(g) = \sum_{i=1}^s g(f_i) \cdot e^\phi(f_i)\, \varphi_j$$

which implies that $\partial_f \mathcal{L}_j(\varphi) \cdot H = \sum_{i=1}^s (g \cdot e^\phi)' \circ f_i \cdot [T_i \cdot H] \cdot \varphi_j$ is differentiable. This finishes the proof of the lemma. $\square$

Proposition 4.11. *Let $r \geq 1$ and $\phi, g \in C^r(M,\mathbb{R})$ be fixed. Then, the map $Diff^r_{loc} \ni f \mapsto \mathcal{L}^n_{f,\phi}(g) \in C^{r-1}(M,\mathbb{R})$ is differentiable. Furthermore, given $H \in C^r(M,M)$, $g_1, g_2 \in C^r(M,\mathbb{R})$ and $t \in \mathbb{R}$ it holds*

i) $D_f(\mathcal{L}^n_{f,\phi}(g))_{|f_0} \cdot H = \sum_{i=1}^n \mathcal{L}^{i-1}_{f_0,\phi}(D_f \mathcal{L}_{f,\phi}(\mathcal{L}^{n-i}_{f_0,\phi}(g))_{|f_0} \cdot H)$;

ii) *there exists $c_{f,\phi} > 0$ so that $\|D_f \mathcal{L}_{f,\phi}(g)_{|f_0} \cdot H\|_0 \leq c_{f,\phi} \|g\|_1 \, \|H\|_1$;*

iii) $D_f \mathcal{L}^n_{f,\phi}(g_1 + tg_2)_{|f_0} \cdot H = D_f \mathcal{L}^n_{f,\phi}(g_1)_{|f_0} \cdot H + t D_f \mathcal{L}^n_{f,\phi}(g_2)_{|f_0} \cdot H$;

iv) *if $\phi \equiv 0$, then $D_f \mathcal{L}^n_f(1)_{|f_0} \cdot H \equiv 0$.*

Proof. For notational simplicity set $\mathcal{L}_f = \mathcal{L}_{f,\phi}$. Then i) is obtained by induction. The case $n = 1$ follows from the previous proposition. Now suppose the formula is valid for n, then using the induction assumption

$$\begin{aligned}
\mathcal{L}^{n+1}_{f_0+H}(g) &= \mathcal{L}_{f_0}(\mathcal{L}^n_{f_0+H}(g)) + D_f \mathcal{L}_f|_{f_0}(\mathcal{L}^n_{f_0+H}(g)) \cdot H + o(H) \\
&= \mathcal{L}_{f_0}\Big(\mathcal{L}^n_{f_0}(g) + \sum_{i=1}^n \mathcal{L}^{i-1}_{f_0}(D_f \mathcal{L}_f(\mathcal{L}^{n-i}_{f_0}(g))_{|f_0} \cdot H) + \hat{o}(H)\Big) \\
&+ D_f \mathcal{L}_f|_{f_0}(\mathcal{L}^n_{f_0+H}(g)) \cdot H + o(H) \\
&= \mathcal{L}_{f_0}\Big(\mathcal{L}^n_{f_0}(g) + \sum_{i=1}^n \mathcal{L}^{i-1}_{f_0}(D_f \mathcal{L}_f(\mathcal{L}^{n-i}_{f_0}(g))_{|f_0} \cdot H) + \hat{o}(H)\Big) \\
&+ D_f \mathcal{L}_f|_{f_0}(\mathcal{L}^n_{f_0}(g)) \cdot H + D_f \mathcal{L}_f|_{f_0}(D_f(\mathcal{L}^n_f(g))_{|f_0} \cdot H + o(H)) \cdot H + \hat{o}(H) \\
&= \mathcal{L}_{f_0}(\mathcal{L}^n_{f_0}(g) + \sum_{i=1}^{n+1} \mathcal{L}^{i-1}_{f_0}(D_f \mathcal{L}_f(\mathcal{L}^{(n+1)-i}_{f_0}(g))_{|f_0} \cdot H) + \tilde{o}(H),
\end{aligned}$$

where $o(H), \hat{o}(H), \tilde{o}(H)$ are terms converging to zero faster than $\|H\|$. This finishes the proof of i). Part ii) is obtained by straightforward computation using the explicit formula from the previous proposition, while part iii) follows using that

$$\begin{aligned}
\mathcal{L}^n_{f+H}(g_1 + tg_2) &= \mathcal{L}^n_{f+H}(g_1) + t\mathcal{L}^n_{f+H}(g_2) \\
&= \mathcal{L}^n_f(g_1) + t\mathcal{L}^n_f(g_2) + D_f \mathcal{L}^n_f(g_1) \cdot H \\
&+ t D_f \mathcal{L}^n_f(g_2) \cdot H + r_1(H) + r_2(H)
\end{aligned}$$

for $r_1(H), r_2(H)$ that tend to zero as H aproaches zero. Finally, part iv) follows immediately from the fact that $\mathcal{L}^n_f(1) \equiv \deg(f)^n$ and that $\deg(f)$ is locally constant. This finishes the proof. $\square$

Throughout, f will denote a local diffeomorphism satisfying (H1) and (H2) and ϕ a Hölder potential such that (P) holds.

Lemma 4.12. *For any probability measure η, the topological pressure $P_{\text{top}}(f,\phi)$ is given by*

$$P_{\text{top}}(f,\phi) = \lim_{n\to+\infty} \frac{1}{n} \log \left[\int \mathcal{L}^n_{f,\phi}(1)\, d\eta\right].$$

In particular, for any given $x \in M$

$$P_{\text{top}}(f,\phi) = \lim_{n\to+\infty} \frac{1}{n} \log[\mathcal{L}^n_{f,\phi}(1)(x)].$$

Proof. Since the second assertion above is a direct consequence of the former one with $\eta = \delta_x$ the Dirac measure at x it is enough to prove the first one. Let η be any fixed probability measure. Recall that the topological pressure is the logarithm of the spectral radius of the transfer operator $\mathcal{L}_{f,\phi}$, that is, $P_{\text{top}}(f,\phi) = \log \lambda_{f,\phi}$.

Moreover, since $\mathcal{L}_{f,\phi}$ is a positive operator then the spectral radius can be computed as

$$\lambda_{f,\phi} = \lim_{n\to+\infty} \sqrt[n]{\|\mathcal{L}^n_{f,\phi}\|} = \lim_{n\to+\infty} \sqrt[n]{\|\mathcal{L}^n_{f,\phi}(1)\|_0}$$

Using that the functions $\lambda^{-n}_{f,\phi}\,\mathcal{L}^n_{\phi,f}(1)$ are uniformly convergent to the eigenfunction $h_{f,\phi}$ which is bounded away from zero and infinity one has that there exists $K>0$ and $n_0 \geq 1$ such that $K^{-1} \leq \lambda^{-n}_{f,\phi}\,\mathcal{L}^n_{\phi,f}(1) \leq K$ for all $n \geq n_0$. In consequence, we get

$$\lim_{n\to\infty} \frac{1}{n} \log \int \lambda^{-n}_{f,\phi}\,\mathcal{L}^n_{\phi,f}(1)\,d\eta = 0,$$

which proves the lemma. □

The next lemma will be fundamental to study the differentiability of equilibrium states. In fact we show that the topological pressure is differentiable once that one requires smooth potentials.

Lemma 4.13 (Differentiability of Topological Pressure with respect to dynamics)**.** *Let ϕ be a fixed $C^{1+\alpha}$ potential on M satisfying (P'). Then the topological pressure function $P_\phi : \mathcal{F}^{1+\alpha}(M) \to \mathbb{R}$ given by $P_\phi(f) = P_{\mathrm{top}}(f,\phi)$ is differentiable with respect to f.*

Proof. As ϕ is considered fixed, for simplicity, we will write ν_f for the conformal measure $\nu_{f,\phi}$ associated to f and the potential ϕ, and we will also use $\mathcal{L}_f$ for the transfer operator $\mathcal{L}_{f,\phi}$. By the last lemma we are reduced to prove the differentiability of the function

$$f \mapsto P(f,\phi) = \lim_{n\to+\infty} \frac{1}{n} \log \int \mathcal{L}^n_f(1)\,d\nu_{f_0}$$

for some fixed f_0. We will use derivation of sequence $P_n(f) = \frac{1}{n}\log\int \mathcal{L}^n_f(1)\,d\nu_f$, which converge to the topological pressure of f uniformly in a small neighborhood of f. By the chain rule, the derivative of P_n with respect to f is given by

$$D_f P_n(f) = \frac{dP_n}{df}(f) = \frac{\nu_{f_0}((\frac{d}{df}\mathcal{L}^n_f(1)(\cdot))}{n\,\nu_{f_0}(\mathcal{L}^n_f(1)(\cdot)}.$$

This yields that

$$\begin{aligned} D_f P_n(\hat f)\cdot(H) &= \frac{\int D_f\mathcal{L}^n_f(1)_{|\hat f}\cdot(H)d\nu_{f_0}}{n\cdot\int \mathcal{L}^n_{\hat f}(1)d\nu_{f_0}} \\ &= \frac{\int \sum_{i=1}^n \mathcal{L}^{i-1}_{\hat f}(D_f\mathcal{L}_f(\mathcal{L}^{n-i}_{\hat f}(1))_{|\hat f}\cdot(H))d\nu_{f_0}}{n\cdot\int \mathcal{L}^n_{\hat f}(1)d\nu_{f_0}}. \end{aligned}$$

In fact the later can be written also as the sum

$$\frac{\int \sum_{i=1}^n \sum_{j=1}^{\deg(\hat f)} e^{\phi(\hat f_j(\cdot))}\cdot D\mathcal{L}^{n-i}_{\hat f}(1)_{|\hat f_j(\cdot)}\cdot[(T_{j|\hat f}\cdot H)(\cdot)])\,d\nu_{f_0}}{n\cdot\int \mathcal{L}^n_{\hat f}(1)d\nu_{f_0}}$$

$$+\frac{\int \sum_{i=1}^n \mathcal{L}^{i-1}_{\hat f}(\sum_{j=1}^{deg(\hat f)} e^{\phi(\hat f_j(\cdot))}\cdot\mathcal{L}^{n-i}_{\hat f}(1)(\hat f_j(\cdot))\cdot D\phi_{|\hat f_j(\cdot)}\cdot[(T_{j|\hat f}\cdot H)(\hat f_j(\cdot))])\,d\nu_{f_0}}{n\cdot\int \mathcal{L}^n_{\hat f}(1)d\nu_{f_0}}. \tag{4.5}$$

To analyze the previous expressions we consider the three sums below

$$B_n(\hat f)\cdot H = \frac{1}{n}\int \sum_{i=1}^{n} \tilde{\mathcal{L}}_{\hat f}^{i-1}\big(\sum_{j=1}^{\deg(\hat f)} e^{\phi(\hat f_j(\cdot))}\cdot D\tilde{\mathcal{L}}_{\hat f}^{n-i}(1)_{|\hat f_j(\cdot)}\cdot [(T_{j|\hat f}\cdot H)(\cdot)]\big)\, d\nu_{f_0}$$

and

$$C_n(\hat f)\cdot H = \frac{1}{n}\int \sum_{i=1}^{n} \tilde{\mathcal{L}}_{\hat f}^{i-1}\Big(\sum_{j=1}^{\deg(\hat f)} e^{\phi(\hat f_j(\cdot))}\tilde{\mathcal{L}}_{\hat f}^{n-i}(1)(\hat f_j(\cdot)) D\phi_{|\hat f_j(\cdot)}[(T_{j|\hat f}\cdot H)(\hat f_j(\cdot))]\Big) d\nu_{f_0}.$$

To establish our result we will use the following:

Claim 1: $B_n(\hat f)\cdot H$ is uniformly convergent on $(\hat f,\phi)$ and $H\in C^{1+\alpha}(M,M)$ with $\|H\|_{1+\alpha}\le 1$ to the series $\int \sum_{j=1}^{\deg(\hat f)} e^{\phi(\hat f_j(\cdot))}\cdot Dh_{\hat f,\hat\phi|\hat f_j(\cdot)}\cdot [(T_{j|\hat f}\cdot H_1)(\cdot)] d\nu_{\hat f}\cdot \int h_{\hat f} d\nu_{f_0}$.

Claim 2: $C_n(\hat f)\cdot H$ is uniformly convergent on $(\hat f,\phi)$ and $H\in C^{1+\alpha}(M,M)$ with $\|H\|_{1+\alpha}\le 1$ to the expression

$$\int \sum_{j=1}^{\deg(\hat f)} e^{\phi(\hat f_j(\cdot))}\cdot h_{\hat f,\phi}(\hat f_j(\cdot))\cdot D\phi_{|\hat f_j(\cdot)}\cdot [(T_{j|\hat f}\cdot H)(\hat f_j(\cdot))]\, d\nu_{\hat f}\cdot \int h_{\hat f,\phi} d\nu_{f_0}.$$

We notice that our result will be a direct consequence of the two claims above. Indeed, using (4.5) it follows that

$$DP_n(\hat f)\cdot (H) = \frac{B_n(\hat f,\hat\phi)\cdot H}{\lambda_{\hat f}\int \tilde{\mathcal{L}}_{\hat f}^{n}(1)d\nu_{f_0}} + \frac{C_n(\hat f,\hat\phi)\cdot H}{\lambda_{\hat f}\int \tilde{\mathcal{L}}_{\hat f}^{n}(1)d\nu_{f_0}}.$$

Moreover, using that $\int \tilde{\mathcal{L}}_{\hat f}^{n}(1)d\nu_{f_0}$ converges to $\int h_{\hat f,\phi}d\nu_{f_0}$ and the uniform limits given by Claims 1 and 2 are we obtain that $DP_n(\hat f)\cdot H$ is uniformly convergent on $(\hat f,\phi)$ and $H\in C^{1+\alpha}(M,M)$ such that $\|H\|=1$ to the sum

$$\lambda_{\hat f,\phi}^{-1}\int \sum_{j=1}^{\deg(\hat f)} e^{\phi(\hat f_j(\cdot))}\cdot Dh_{\hat f\phi|\hat f_j(\cdot)}\cdot [(T_{j|\hat f}\cdot H_1)(\cdot)]\, d\nu_{\hat f}$$
$$+\lambda_{\hat f,\phi}^{-1}\int \sum_{j=1}^{\deg(\hat f)} e^{\phi(\hat f_j(\cdot))}\cdot h_{\hat f,\phi}(\hat f_j(\cdot))\cdot D\phi_{|\hat f_j(\cdot)}\cdot [(T_{j|\hat f}\cdot H)(\hat f_j(\cdot))]\, d\nu_{\hat f}.$$

Hence

$$D_f\lambda_{f,\phi\ |f_0,\phi}\cdot H = \sum_{j=1}^{\deg(f_0)}\int e^{\phi_0(f_{0,j}(\cdot))} Dh_{f_0,\phi_0|f_{0,j}(\cdot)}[(T_{j|f_0}\cdot H)(\cdot)]\, d\nu_{f_0,\phi_0}$$
$$+\sum_{j=1}^{deg(f_0)}\int e^{\phi(f_{0,j}(\cdot))} h_{f_0,\phi}(f_{0,j}(\cdot)) D\phi_{|f_{0,j}(\cdot)}[(T_{j|f_0}\cdot H)(f_{0,j}(\cdot))] d\nu_{f_0}.$$

which proves the lemma. Therefore, in the remaining we prove the previous claims. First we prove Claim 1. Observe that the following uniform convergence holds

$$\frac{1}{n}\sum_{i=0}^{n-1}\int\sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))}\cdot D\tilde{\mathcal{L}}^i_{\hat{f}}(1)_{|\hat{f}_j(\cdot)}\cdot[(T_{j|\hat{f}}\cdot H)(\cdot)]\,d\nu_{\hat{f}}\cdot\int h_{\hat{f},\phi}d\nu_{f_0}$$

$$\xrightarrow[n\to\infty]{}\int\sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))}\cdot Dh_{\hat{f},\phi|\hat{f}_j(\cdot)}\cdot[(T_{j|\hat{f}}\cdot H)(\cdot)]d\nu_{\hat{f},\hat{\phi}}\cdot\int h_{\hat{f},\phi}d\nu_{f_0},$$

Moreover, one also has that

$$\begin{aligned}
&\Big|B_n(\hat{f},\hat{\phi})\cdot H-\frac{1}{n}\sum_{i=0}^{n-1}\int\sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))}D\tilde{\mathcal{L}}^i_{\hat{f},\phi}(1)_{|\hat{f}_j(\cdot)}[(T_{j|\hat{f}}\cdot H)(\cdot)]\,d\nu_{\hat{f}}\cdot\int h_{\hat{f},\phi}d\nu_{f_0}\Big|\\
&\quad\le\frac{1}{n}\sum_{i=1}^{n}\Big|\int\sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))}D\tilde{\mathcal{L}}^{n-i}_{\hat{f}}(1)_{|\hat{f}_j(\cdot)}[(T_{j|\hat{f}}\cdot H)(\cdot)]\,d(\tilde{\mathcal{L}}^{*i-1}_{\hat{f}}\nu_{f_0})\\
&\quad-\int\sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))}D\tilde{\mathcal{L}}^{n-i}_{\hat{f}}(1)_{|\hat{f}_j(\cdot)}[(T_{j|\hat{f}}\cdot H)(\cdot)]\,d\nu_{\hat{f}}\cdot\int h_{\hat{f},\phi}d\nu_{f_0}\Big|\\
&\quad\le\frac{1}{n}\sum_{i=1}^{n}k\tau^{i-1}\deg(\hat{f})\|e^{\phi}\|_0\max_{1\le j\le\deg(\hat{f})}\{\|(T_{j|\hat{f}}\cdot H)\|_0\}\,\|\tilde{\mathcal{L}}^{n-i}_{\hat{f}}(1)\|_1\\
&\quad\le\frac{1}{n}\sum_{i=1}^{n}k\tau^{i-1}\deg(\hat{f})\|e^{\phi}\|_0\max_{1\le j\le\deg(\hat{f})}\{\|(T_{j|\hat{f}}\cdot H)\|_0\}[k\tau^{n-i}+\|h_{\hat{f},\phi}\|_1]
\end{aligned}$$

which is uniformly convergent to zero with respect to $(\hat{f},\phi)$ and all $H\in C^{1+\alpha}(M,M)$ with $\|H\|_{1+\alpha}\le 1$. This proves Claim 1. We now proceed to prove Claim 2.

$$\frac{1}{n}\sum_{i=0}^{n-1}\int\sum_{j=1}^{deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))}\tilde{\mathcal{L}}^i_{\hat{f}}(1)(\hat{f}_j(\cdot))D\phi_{|\hat{f}_j(\cdot)}[(T_{j|\hat{f}}\cdot H)(\hat{f}_j(\cdot))]d\nu_{\hat{f},\phi}\cdot\int h_{\hat{f},\phi}d\nu_{f_0}$$

$$\xrightarrow[n\to+\infty]{}\int\sum_{j=1}^{deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))}h_{\hat{f},\phi}(\hat{f}_j(\cdot))D\phi_{|\hat{f}_j(\cdot)}\cdot[(T_{j|\hat{f}}\cdot H)(\hat{f}_j(\cdot))]d\nu_{\hat{f}}\cdot\int h_{\hat{f},\phi}d\nu_{f_0},$$

uniformly with respect to $(\hat{f},\phi)$ and H. We have that

$$\Big|C_n(\hat{f})\cdot H-\frac{1}{n}\sum_{i=0}^{n-1}\int\sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))}\tilde{\mathcal{L}}^i_{\hat{f}}(1)(\hat{f}_j(\cdot))D\phi_{|\hat{f}_j(\cdot)}[(T_{j|\hat{f}}\cdot H)(\hat{f}_j(\cdot))]d\nu_{\hat{f}}\int h_{\hat{f},\phi}d\nu_{f_0}\Big|$$

$$\leq \frac{1}{n}\sum_{i=1}^{n}\Big|\int\sum_{j=1}^{deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))}\tilde{\mathcal{L}}^{n-i}_{\hat{f}}(1)(\hat{f}_j(\cdot))D\phi_{|\hat{f}_j(\cdot)}[(T_{j|\hat{f}}\cdot H)(\hat{f}_j(\cdot))]\;d(\tilde{\mathcal{L}}^{*i-1}_{\hat{f}}\nu_{f_0})$$

$$-\int\sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))}\tilde{\mathcal{L}}^{n-i}_{\hat{f}}(1)(\hat{f}_j(\cdot))D\phi_{|\hat{f}_j(\cdot)}[(T_{j|\hat{f}}\cdot H)(\hat{f}_j(\cdot))]\;d\nu_{\hat{f},\phi}\cdot\int h_{\hat{f},\hat{\phi}}d\nu_{f_0}$$

$$\leq \frac{1}{n}\sum_{i=1}^{n}k\tau^{i-1}\|\sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))}\tilde{\mathcal{L}}^{n-i}_{\hat{f}}(1)(\hat{f}_j(\cdot))\cdot D\phi_{|\hat{f}_j(\cdot)}[(T_{j|\hat{f}}\cdot H)(\hat{f}_j(\cdot))]\|_0$$

$$\leq \frac{1}{n}\sum_{i=1}^{n}Ck\tau^{i-1}\deg(\hat{f})\cdot\|e^{\phi}\|_0\cdot\|\tilde{\mathcal{L}}^{n-i}_{\hat{f}}(1)\|_0$$

$$\leq \frac{1}{n}\sum_{i=1}^{n}Ck\tau^{i-1}\deg(\hat{f})\cdot\|e^{\phi}\|_0\cdot(k\tau^{n-i}+\|h_{\hat{f},\phi}\|_0)$$

where $C=\|\phi\|_1\cdot\max_{j=1,\dots,\deg(\hat{f})}\{\|[(T_{j|\hat{f}}\cdot H)(\hat{f}_j(\cdot))]\|_0\}$. Since the later expression is uniformly convergent to zero with respect to $(\hat{f},\phi)$ and $H\in C^{1+\alpha}(M,M)$ such that $\|H\|_{1+\alpha}\leq 1$ this proves Claim 2 and finishes the proof of the lemma. □

Corollary 4.14. *The topological pressure $P_{top}:\mathcal{F}^{1+\alpha}\times\mathcal{W}^{1+\alpha}\to\mathbb{R}$ is differentiable.*

Proof. Just note that the derivatives calculated in Corollary 4.5 and in the Lemma above are partial derivatives for the function $P_{top}(f,\phi)$, and jointly continuous with respect to both variables f and ϕ.

□

4.3. **Differentiability of maximal entropy measure with respect to dynamics.** In this section we deduce the differentiable dependence of the thermodynamical quantities on the dynamical system. Through this section we deal with maximal entropy measures and henceforth we fix the potential $\phi\equiv 0$ and fix f_0 local diffeomorphism satisfying (H1) and (H2). Recall that for every C^1 local diffeomorphism f satisfying (H1) and (H2) we have maximal eigenvalue $\lambda_f=\deg(f)$, eigenfunction $h_f=\frac{d\mu_f}{d\nu_f}=1$ and conformal measure $\nu_f=\mu_f$ for the Perron-Frobenius operator. In particular, the entropy $h_{\text{top}}(f)=\log\deg(f)$ is constant. Now we can establish that the transfer operators are differentiable with respect to the potential ϕ and the dynamical system f.

Let $r\in\mathbb{N}_0$ and $\alpha\in[0,1)$ be such that $r+\alpha>0$. It follows from Theorems 3.3 and 3.5 that all transfer operators $\mathcal{L}_f: C^{k+\alpha}(M,\mathbb{R})\to C^{k+\alpha}(M,\mathbb{R})$ have the spectral gap property for $k+\alpha\in\{\alpha,1+\alpha,\dots,r+\alpha\}\cap\mathbb{R}^*_+$, provided that f is sufficiently $C^{r+\alpha}$-close to f_0. In consequence, it is not hard to check that if $E^{k+\alpha}_{0,f}=\{g\in C^{k+\alpha}(M,\mathbb{R}):\int g\,d\nu_f=0\}$ then $C^{k+\alpha}(M,\mathbb{R})=\{\ell\,h_f:\ell\in\mathbb{R}\}\oplus E^{k+\alpha}_{0,f}$ is a $\tilde{\mathcal{L}}_f$-invariant decomposition in $C^{k+\alpha}(M,\mathbb{R})$. Furthermore there are constants

$C_{f,k+\alpha} > 0$ and $\tau_{f,k+\alpha} \in (0,1)$ such that for all $g \in E^{k+\alpha}_{0,f}$ it follows:

$$\|\mathcal{L}^n_f g\|_0 \leq C_{f,k+\alpha} \tau^n_{f,k+\alpha} \|g\|_0, \quad \text{for all } n \geq 1.$$

Set $C_f = \max\{C_{f,k+\alpha} : k \leq r\}$, $\tau_f = \max\{\tau_{f,k+\alpha} : k \leq r\}$ and recall that these can be assumed uniform in a neighborhood of f_0. Consider also the spectral projection $P^{k+\alpha}_{0,f} : C^{k+\alpha}(M,\mathbb{R}) \to E^{k+\alpha}_{0,f}$ given by $P^{k+\alpha}_{0,f}(g) = g - \int g \, d\nu_f$. In what follows, when no confusion is possible we shall omit the dependence on f in the corresponding subspaces and spectral projections.

Theorem 4.15. *For any $g \in C^{r+\alpha}(M,\mathbb{R})$ the map $\mathcal{F}^{r+\alpha} \ni f \mapsto \int g \, d\mu_f$ is C^1-differentiable and its derivative acting in $H \in C^{r+\alpha}(M,M)$ is given by*

$$D_f \mu_f(g)_{|f_0} \cdot H = \sum_{i=0}^{\infty} \int D_f \tilde{\mathcal{L}}_f (\, \tilde{\mathcal{L}}^i_{f_0}(P_0(g)) \,) \cdot H \, d\mu_{f_0}.$$

Proof. Let $g \in C^{r+\alpha}(M,\mathbb{R})$ and $f_0 \in \mathcal{F}^{r+\alpha}$ be fixed. We define a sequence of maps $F_n : \mathcal{F}^{r+\alpha} \to \mathbb{R}$ given by $F_n(f) = \int \tilde{\mathcal{L}}^n_f(g) \, d\mu_{f_0}$ and notice that $F_n(f)$ is convergent to $\int g \, d\mu_f$, whereas the convergence is uniform in a sufficiently small neighborhood of f_0. Moreover, if $H \in C^{r+\alpha}(M,M)$ then

$$\begin{aligned}
DF_n(f) \cdot H &= \sum_{i=1}^{n-1} \int \tilde{\mathcal{L}}^{i-1}_f (D_f \tilde{\mathcal{L}}_f (\tilde{\mathcal{L}}^{n-i}_f(g)) \ \cdot H) \, d\mu_{f_0} \\
&= \sum_{i=1}^{n-1} \int \tilde{\mathcal{L}}^{i-1}_f (D_f \tilde{\mathcal{L}}_f \left(\int g \, d\mu_f + \tilde{\mathcal{L}}^{n-i}_f(P_0(g)) \right) \ \cdot H) \, d\mu_{f_0} \\
&= \sum_{i=1}^{n-1} \int \tilde{\mathcal{L}}^{i-1}_f (D_f \tilde{\mathcal{L}}_f (\tilde{\mathcal{L}}^{n-i}_f(P_0(g))) \ \cdot H) \, d\mu_{f_0}.
\end{aligned}$$

On the other hand, since we assumed $\phi \equiv 0$ then $\mu_f = \nu_f$ and $\mathcal{L}^*_f \mu_f = \mu_f$. Thus,

$$\sum_{i=1}^{n-1} \int D_f \tilde{\mathcal{L}}_f (\tilde{\mathcal{L}}^{n-i}_f(P_0(g)) \ \cdot H \, d\mu_f = \sum_{i=0}^{n-1} \int D_f \tilde{\mathcal{L}}_f (\tilde{\mathcal{L}}^{n-i}_f(P_0(g)) \ \cdot H \, d\mu_f$$

and

$$\begin{aligned}
\sum_{i=0}^{n-1} \Big| \int D_f \tilde{\mathcal{L}}_f (\tilde{\mathcal{L}}^i_f(P_{0,f}(g))) \ \cdot H \, d\mu_f \Big| &\leq \sum_{i=0}^{n-1} \|D_f \tilde{\mathcal{L}}_f (\tilde{\mathcal{L}}^i_f(P_{0,f}(g)))_f \ \cdot H\|_0 \\
&\leq \sum_{i=0}^{n-1} \|\tilde{\mathcal{L}}^i_f(P_{0,f}(g))\|_1 \cdot \|H\|_1 \cdot c_f \\
&\leq \sum_{i=0}^{n-1} C_f \tau^i_f \cdot 2 \cdot \|g\|_1 \cdot c_f \cdot \|H\|_1,
\end{aligned}$$

that is bounded from above by $\frac{C_f}{1-\tau_f} \cdot 2\|g\|_1 c_f \|H\|_1$. Moreover, since C_f, τ_f and c_f may be taken uniform in the neighborhood of f_0 we get the previous upper bound is uniform on the unit sphere of $C^{r+\alpha}(M,M)$. This implies that the limit

$$\lim_{n\to+\infty} \sum_{i=1}^{n-1} \int D_f \tilde{\mathcal{L}}_f (\tilde{\mathcal{L}}^{n-i}_f(P_0(g))) \cdot H \, d\mu_f,$$

does exist and is uniform with respect to the dynamics and the unit sphere of $C^{r+\alpha}(M,M)$. We proceed and estimate

$$
\begin{aligned}
&|DF_n(f)\cdot H-\sum_{i=1}^{n-1}\int D_f\tilde{\mathcal{L}}_f(\tilde{\mathcal{L}}_f^{n-i}(P_0(g)))\cdot H\,d\mu_f| \\
&=\left|\sum_{i=1}^{n-1}\left[\int \tilde{\mathcal{L}}_f^{i-1}(D_f\tilde{\mathcal{L}}_f(\tilde{\mathcal{L}}_f^{n-i}(P_0(g)))\cdot H)\,d\mu_{f_0}-\int D_f\tilde{\mathcal{L}}_f(\tilde{\mathcal{L}}_f^{n-i}(P_0(g)))\cdot H\,d\mu_f\right]\right| \\
&=\left|\sum_{i=1}^{n-1}\left[\int D_f\tilde{\mathcal{L}}_f(\tilde{\mathcal{L}}_f^{n-i}(P_0(g)))\cdot H\,d\tilde{\mathcal{L}}_f^{*i-1}(\mu_{f_0})-\int D_f\tilde{\mathcal{L}}_f(\tilde{\mathcal{L}}_f^{n-i}(P_0(g)))\cdot H\,d\mu_f\right]\right| \\
&\le\sum_{i=1}^{n-1}C_f\tau_f^{i-1}2\|D_f\tilde{\mathcal{L}}_f(\tilde{\mathcal{L}}_f^{n-i}(P_0(g)))H\|_0\le\sum_{i=1}^{n-1}C_f\tau_f^{i-1}\cdot 2\|\tilde{\mathcal{L}}_f^{n-i}(P_0(g))\|_1c_f\|H\|_1 \\
&\le\sum_{i=1}^{n-1}C_f\tau_f^{i-1}\cdot 2\cdot C_f\tau_f^{n-i}\cdot 2\cdot\|g\|_1\cdot c_f\|H\|_1\le 4c_fC_f^2\,(n-1)\tau^{n-1}\cdot\|g\|_1\cdot\|H\|_1
\end{aligned}
$$

which converges to zero. Thus $\lim DF_n(f)\cdot H=\sum_{i=1}^{n-1}\int D_f\tilde{\mathcal{L}}_f(\tilde{\mathcal{L}}_f^{n-i}(P_0(g)))\cdot H\,d\mu_f$ uniformly with respect to the dynamics f and the unit sphere of $C^{r+\alpha}(M,M)$. One can deduce that for all f close to f_0 the sequence $F_n(f)$ converges uniformly to $\int g\,d\mu_f$ and the sequence DF_n is also uniformly convergent to the linear continuous functional defined above. We conclude that

$$D_f\mu_f(g)_{|f_0}\cdot H=\sum_{i=0}^{\infty}\int D_f\tilde{\mathcal{L}}_f(\tilde{\mathcal{L}}_{f_0}^{i}(P_0(g)))\cdot H d\mu_{f_0}.$$

This finishes the proof of the theorem. □

At this point we have proven Theorems A to D. Concerning the differentiability results it remains to prove Corollary B. This is a consequence of the following auxiliary lemma.

Lemma 4.16. *Let X and Λ be metric spaces such that X is compact and $(\eta_\lambda)_{\lambda\in\Lambda}$ is a collection of probability measures in X so that $\Lambda\ni\lambda\mapsto\eta_\lambda$ is continuous in the weak* topology. If $\lambda\mapsto\psi(\lambda)\in C^0(X,\mathbb{R})$ is a continuous mapping in the supremum norm then $\Lambda\ni\lambda\mapsto\int\psi(\lambda)\,d\eta_\lambda$ is a continuous transformation.*

Proof. Let $\lambda_0\in\Lambda$ be arbitrary and fixed. Given $\lambda\in\Lambda$, using the triangular inequality we can write

$$
\begin{aligned}
\left|\int\psi(\lambda)\,d\eta_\lambda-\int\psi(\lambda_0)\,d\eta_{\lambda_0}\right| &\le\left|\int\psi(\lambda)\,d\eta_\lambda-\int\psi(\lambda_0)\,d\eta_\lambda\right| \\
&+\left|\int\psi(\lambda_0)\,d\eta_\lambda-\int\psi(\lambda_0)\,d\eta_{\lambda_0}\right| \\
&\le\|\psi(\lambda)-\psi(\lambda_0)\|_0+\left|\int\psi(\lambda_0)\,d\eta_\lambda-\int\psi(\lambda_0)\,d\eta_{\lambda_0}\right|,
\end{aligned}
$$

which can be taken arbitrary small provided that λ is close to λ_0. This finishes the proof of the lemma. □

5. Free energy and large deviations

In this section we prove the differentiability of the free energy function and deduce some further properties for large deviations.

5.1. **Free energy function.** First we establish some properties of the free energy function as consequence of the spectral gap property. Recall that a potential $\psi : M \to \mathbb{R}$ is *cohomologous to a constant* if there exists $A \in \mathbb{R}$ and a potential $\tilde{\psi} : M \to \mathbb{R}$ such that $\psi = \tilde{\psi} \circ f - \tilde{\psi} + A$. Now we prove the following:

Proposition 5.1. *Let f and ϕ satisfy assumptions (H1), (H2) and (P). Then for any Hölder observable $\psi : M \to \mathbb{R}$ there exists $t_\psi > 0$ such that for all $|t| \le t_\psi$ the following limit exists*

$$\mathcal{E}_{f,\psi}(t) := \lim_{n\to\infty} \frac{1}{n} \log \int e^{tS_n\psi}\, d\mu_\phi = P_{\text{top}}(f, \phi + t\psi) - P_{\text{top}}(f, \phi).$$

Moreover, if ψ is cohomologous to a constant then $t \mapsto \mathcal{E}_{f,\psi}(t)$ is linear and, otherwise, $t \mapsto \mathcal{E}_{f,\psi}(t)$ is real analytic, strictly convex. Furthermore, if (f, ϕ) satisfy $(H1), (H2), (P')$, then for every fixed $t \in \mathbb{R}$ the function $\mathcal{F}^{1+\alpha} \ni f \mapsto \mathcal{E}_{f,\psi}(t)$ is differentiable and $\mathcal{F}^{1+\alpha} \ni f \mapsto \mathcal{E}'_{f,\psi}(t)$ is continuous.

Proof. This proof goes along some well known arguments that we include here for completeness. Observe first that for all $n \in \mathbb{N}$

$$\begin{aligned}\int e^{tS_n\psi}\, d\mu_\phi &= \int \lambda_{f,\phi}^{-n}\, \mathcal{L}_{f,\phi}^n(h_\phi e^{tS_n\psi})\, d\nu_\phi \\ &= \left(\frac{\lambda_{f,\phi+t\psi}}{\lambda_{f,\phi}}\right)^n \int \lambda_{f,\phi+t\psi}^{-n}\, \mathcal{L}_{f,\phi+t\psi}^n(h_{f,\phi})\, d\nu_\phi.\end{aligned}$$

Since (P) is an open condition, then for every $|t| \le t_\psi$ the potential $\phi + t\psi$ satisfies (P) provided that t_ψ is small enough. Since $h_{f,\phi}$ is positive and bounded away from zero and infinity this implies that $\lambda_{f,\phi+t\psi}^{-n}\, \mathcal{L}_{f,\phi+t\psi}^n(h_{f,\phi})$ is uniformly convergent to $h_{f,\phi+t\psi}$, thus uniformly bounded from zero and infinity for all large n. Therefore using the dominated convergence theorem

$$\lim_{n\to\infty} \frac{1}{n} \log \int e^{tS_n\psi}\, d\mu_\phi = \log \lambda_{f,\phi+t\psi} - \log \lambda_{f,\phi} = P_{\text{top}}(f, \phi + t\psi) - P_{\text{top}}(f, \phi),$$

proving the first assertion of the proposition. Now, assume first that there exists $A \in \mathbb{R}$ and a potential $\tilde{\psi} : M \to \mathbb{R}$ such that $\psi = \tilde{\psi} \circ f - \tilde{\psi} + A$. Then it follows from the variational principle and invariance that

$$\begin{aligned}P_{\text{top}}(f, \phi + t\psi) &= \sup_{\mu \in \mathcal{M}_1(f)} \left\{ h_\mu(f) + \int [\phi + t\psi]\, d\mu \right\} \\ &= tA + \sup_{\mu \in \mathcal{M}_1(f)} \left\{ h_\mu(f) + \int \phi\, d\mu \right\} \\ &= tA + P_{\text{top}}(f, \phi)\end{aligned}$$

and, consequently, $\mathcal{E}_{f,\psi}(t) = tA$ is linear.

Now, it remains to prove that if ψ is not cohomologous to a constant then the free energy function is strictly convex. Since $t \mapsto P_{\text{top}}(f, \phi + t\psi)$ is real analytic (recall Remark 4.2) then to prove that $t \mapsto \mathcal{E}_{f,\psi}(t)$ is strictly convex it is enough to show that $\mathcal{E}''_{f,\psi}(t) > 0$ for all t. Assume that there exists t such that $\mathcal{E}''_{f,\psi}(t) = 0$. Up to replace ϕ by the potential $\tilde{\phi} = \phi + t\psi$ we may assume without loss of generality

that $t=0$, that is, $\mathcal{E}''_{f,\psi}(0)=0$. Hence, using Corollary 4.5 and differentiation under the sign of integral we obtain

$$\mathcal{E}'_{f,\psi}(t)=\int \psi\, d\mu_{\phi+t\psi}=\lim_{n\to\infty}\frac{1}{n}\frac{\int (S_n\psi)\, e^{tS_n\psi} d\mu_\phi}{\int e^{tS_n\psi} d\mu_\phi}$$

(hence $\mathcal{E}'_{f,\psi}(0)=\int \psi\, d\mu_\phi$). Using Theorem B and differentiating again with respect to t under the sign of integral it follows that

$$\mathcal{E}''_{f,\psi}(t)=\lim_{n\to\infty}\frac{1}{n}\left[\frac{\int (S_n\psi)^2\, e^{tS_n\psi} d\mu_\phi}{\int e^{tS_n\psi} d\mu_\phi}-\left(\frac{\int S_n\psi\, e^{tS_n\psi} d\mu_\phi}{\int e^{tS_n\psi} d\mu_\phi}\right)^2\right]\geq 0$$

(hence $\mathcal{E}''_{f,\psi}(0)=\lim_{n\to\infty}\frac{1}{n}[\int (S_n\psi)^2\, d\mu-(\int S_n\psi\, d\mu)^2]>0$) because, if $\mu_n=e^{tS_n\psi}d\mu_\phi$ the inequality is equivalent to $\int S_n\psi\, d\mu_n\leq(\int (S_n\psi)^2\, d\mu_n)^{\frac12}(\int 1\, d\mu_n)^{\frac12}$ that holds by Hölder's inequality. In particular, $\mathcal{E}''_{f,\psi}(t)=0$ if and only ψ is cohomologous to a constant. Since the topological pressure is differentiable with respect to the dynamics the proof of the proposition is now complete. □

Remark 5.2. We also note that if (f,ϕ) satisfies $(H1),(H2),(P)$, then for every fixed t the functions $\mathcal{F}\ni f\mapsto \mathcal{E}_{f,\psi}(t)$ and $\mathcal{F}\ni f\mapsto \mathcal{E}'_{f,\psi}(t)$ are still continuous.

The following simple lemma illustrates some characteristics of the behaviour on the free energy function and follows directly from the definitions. For that reason we shall omit its proof.

Lemma 5.3. *For any Hölder continuous potential ψ, the free energy function $[-t_\psi,t_\psi]\ni t\to\mathcal{E}_{f,\psi}(t)$ satisfies:*

1. $\mathcal{E}_{f,\psi}(0)=0$;
2. $t\inf\psi\leq\mathcal{E}_{f,\psi}(t)\leq t\sup\psi$ *for all* $t\in(0,t_\psi]$;
3. $t\sup\psi\leq\mathcal{E}_{f,\psi}(t)\leq t\inf\psi$ *for all* $t\in[-t_\psi,0)$.

In what follows assume that ψ is not cohomologous to a constant and that $m=\int\psi\, d\mu_\phi=0$. Therefore, since the function $[-t_\psi,t_\psi]\ni t\to\mathcal{E}_{f,\psi}(t)$ is strictly convex it is well defined the "local" Legendre transform $I_{f,\psi}$ given by

$$I_{f,\psi}(s)=\sup_{-t_\psi\leq t\leq t_\psi}\big\{st-\mathcal{E}_{f,\psi}(t)\big\}.$$

This is a convex function since it is supremum of linear functions and, using that $\mathcal{E}_{f,\psi}$ is strictly convex we get that $I_{f,\psi}\geq 0$. Moreover, by Jensen's inequality one has that $\mathcal{E}_{f,\psi}(t)\geq tm$, that is, $mt-\mathcal{E}_{f,\psi}(t)\leq 0$.

In particular, since the free energy function is differentiable then we have the variational property that $I_{f,\psi}(\mathcal{E}'_{f,\psi}(t))=t\mathcal{E}'_{f,\psi}(t)-\mathcal{E}_{f,\psi}(t)$ and the domain of $I_{f,\psi}$ contains the interval $[\mathcal{E}'_{f,\psi}(-t_\psi),\mathcal{E}'_{f,\psi}(t_\psi)]$. Moreover, $I_{f,\psi}(s)=0$ if and only if $s=m$ belongs to the domain of $I_{f,\psi}$ and m is the minimum of the positive convex function $I_{f,\psi}$. It is also well known that the strict convexity of $\mathcal{E}_{f,\psi}$ together with differentiability of $\mathcal{E}_{f,\psi}$ yields that $[-t_\psi,t_\psi]\ni t\mapsto I_{f,\psi}(t)$ is strictly convex and differentiable. In fact we collect all these results in the following:

Lemma 5.4. *Let $f\in\mathcal{F}$ be arbitrary and let ψ be an Hölder continuous observable. Then the rate function $I_{f,\psi}$ satisfies:*

1. *The domain $[\mathcal{E}'_{f,\psi}(-t_\psi),\mathcal{E}'_{f,\psi}(t_\psi)]$ contains $m_{f,\phi}=\int\psi\, d\mu_{f,\phi}$;*
2. *$I_{f,\psi}\geq 0$ is strictly convex and $I_{f,\psi}(s)=0$ if and only $s=\int\psi\, d\mu_{f,\phi}$;*
3. *$s\mapsto I_{f,\psi}(s)$ is real analytic.*

5.2. **Estimating deviations.** Now we use the previous free energy function to obtain a "local" large deviation results. In fact, the following results hold from Gartner-Ellis theorem (see e.g. [DZ98]) as a consequence of the differentiability of the free energy function.

Theorem 5.5. *Given any interval* $[a,b] \subset [\mathcal{E}'_{f,\psi}(-t_\psi), \mathcal{E}'_{f,\psi}(t_\psi)]$ *it holds that*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_\phi \left(x \in M : \frac{1}{n} S_n\psi(x) \in [a,b] \right) \leq - \inf_{s\in[a,b]} I_{f,\psi}(s)$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_\phi \left(x \in M : \frac{1}{n} S_n\psi(x) \in (a,b) \right) \geq - \inf_{s\in(a,b)} I_{f,\psi}(s)$$

Furthermore, in this context we can deduce extra regularity of the large deviations rate function. In fact, as a consequence of our differentiability results we conclude the following.

Corollary 5.6. *Let* ψ *be a Hölder continuous observable. Then, there exists an interval* $J \subset \mathbb{R}$ *such that for all* $[a,b] \subset J$ *and* $f \in \mathcal{F}^{1+\alpha}$ *it holds that*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_\phi \left(x \in M : \frac{1}{n} S_n\psi(x) \in [a,b] \right) \leq - \inf_{s\in[a,b]} I_{f,\psi}(s)$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_\phi \left(x \in M : \frac{1}{n} S_n\psi(x) \in (a,b) \right) \geq - \inf_{s\in(a,b)} I_{f,\psi}(s)$$

Moreover, if V *is a compact metric space and* $V \ni v \mapsto f_v \in \mathcal{F}$ *is an injective and continuous map then the rate function* $(s,v) \mapsto I_{f_v,\psi}(s)$ *is continuous on* $J \times V$.

Proof. Fix $f_0 \in \mathcal{F}^{1+\alpha}$. We obtain a large deviation principle for Birkhoff averages on subintervals of a given interval $[\mathcal{E}'_{f_0,\psi}(-t_\psi), \mathcal{E}'_{f_0,\psi}(t_\psi)]$ given by Theorem 5.5. Observe that the interval $[\mathcal{E}'_{f,\psi}(-t_\psi), \mathcal{E}'_{f,\psi}(t_\psi)]$ is non-degenerate and varies continuously with $f \in \mathcal{F}^{1+\alpha}$ and ψ. Hence, we may take a non-degenerate interval J contained in all intervals $[\mathcal{E}'_{f,\psi}(-t_\psi), \mathcal{E}'_{f,\psi}(t_\psi)]$ for all $f \in \mathcal{F}^{1+\alpha}$ sufficiently close to f_0. This proves the first assertion above.

Finally, from the variational relation using the Legendre transform and the convexity of the free energy function (that is, $\mathcal{E}''_{f,\psi}(t) > 0$ for all t) we get that for any $s \in J$ there exists a unique $t = t(s,v)$ such that $s = \mathcal{E}'_{f_v,\psi}(t)$ and

$$I_{f_v,\psi}(s) = s \cdot t(s,v) - \mathcal{E}_{f,\psi}(t(s,v)). \tag{5.1}$$

Now, we consider the continuous skew-product

$$\begin{array}{rcl} F : V \times J & \to & V \times \mathbb{R} \\ (v,t) & \mapsto & (v, \mathcal{E}'_{f_v,\psi}(t)) \end{array}$$

and notice that it is injective because it is strictly increasing along the fibers. Since $V \times J$ is a compact metric space then F is a homeomorphism onto its image $F(V \times J)$. In particular this shows that for every $(v,s) \in F(V \times J)$ there exists a unique $t = t(v,s)$ varying continuously with (v,s) such that $F(v, t(v,s)) = (v,s)$ and $s = \mathcal{E}'_{f_v,\psi}(t)$. Finally, relation (5.1) above yields that $(s,v) \mapsto I_{f_v,\psi}(s)$ is continuous on $J \times V$. This finishes the proof of the corollary. □

It is not hard to check that the rate function is real analytic with respect to the potential. However, since the proof is much simpler than the previous one we shall omit it and leave as an exercise to the reader.

6. Examples

In this section we provide some possible applications to the study of thermodynamical formalism and large deviations of some relevant classes of non-uniformly expanding dynamical systems.

Example 6.1. Let $f_0 : \mathbb{T}^d \to \mathbb{T}^d$ be a linear expanding map. Fix some covering $\mathcal{U}$ by domains of injectivity for f_0 and some $U_0 \in \mathcal{U}$ containing a fixed (or periodic) point p. Then deform f_0 on a small neighborhood of p inside U_0 by a pitchfork bifurcation in such a way that p becomes a saddle for the perturbed local diffeomorphism f. In particular, such perturbation can be done in the C^r-topology, for every $r > 0$. By construction, f coincides with f_0 in the complement of P_1, where uniform expansion holds. Observe that we may take the deformation in such a way that f is never too contracting in P_1, which guarantees that conditions (H1) and (H2) hold. Since the later are open conditions let $\mathcal{F}$ a small open neighborhood of f by $C^{1+\alpha}$ local diffeomorphisms satisfying (H1) and (H2). Since condition (P') is clearly satisfied by $\phi \equiv 0$ one can take $\mathcal{W}$ to be an open set of $C^{1+\alpha}$-potentials close to zero and satisfying (P') with uniform constants. It follows from [VV10, CV11] that there exists a unique equilibrium state for f with respect to ϕ, is has exponential decay of correlations in the space of Hölder observables and that equilibrium states and topological pressure vary continuously with the dynamics.

Concerning higher regularity of these functions it follows from Theorems A to D that the pressure function $(f, \phi) \mapsto P(f, \phi)$, the invariant density function $(f, \phi) \mapsto h_{f,\phi}$ and the conformal measure function $(f, \phi) \mapsto \nu_{f,\phi}$ are analytical in ϕ and differentiable with respect to f. Furthermore, the largest and smallest Lyapunov exponents and the metric entropy of the equilibrium states $\mu_{f,\phi}$ vary continuously with respect to f and ϕ. Finally, the unique measure of maximal entropy μ_f is differentiable with respect to f.

Let us mention an easy modification of the previous example allows to consider expanding maps with indifferent periodic points in both one-dimensional or higher-dimensional settings. We discuss in detail a one-dimensional important example.

Example 6.2. *(Manneville-Pomeau map)* Given $\alpha > 0$, let $f_\alpha : [0,1] \to [0,1]$ be the $C^{1+\alpha}$-local diffeomorphism given by

$$f_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } 0 \le x \le \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} < x \le 1 \end{cases} \tag{6.1}$$

and the family of potentials $\varphi_{\alpha,t} = -t \log |Df_\alpha|$. Conditions (H1) and (H2) are clearly verified by f_α. Moreover, it is well known that if $\alpha \in (0,1)$ then an intermitency phenomenon occurs for $t = 1$. This no longer occurs whenever t is close to zero for any $\alpha > 0$ as we now discuss in detail.

Assume first $\alpha \in (0,1]$. The family $\varphi_{\alpha,t}$ of C^α-potentials do satisfy condition (P) for all $|t| \le t_0$ small since

$$|\varphi_{\alpha,t}(x) - \varphi_{\alpha,t}(y)| = |t \log |Df_\alpha(x)| - t \log |Df_\alpha(y)|| = |t| \log \frac{|Df_\alpha(x)|}{|Df_\alpha(y)|} \le |t| \log(2+\alpha)$$

Hence, it follows from [CV11] that for all $|t| \le t_0$ there exists a unique equilibrium state $\mu_{\alpha,t}$ for f_α with respect to $\varphi_{\alpha,t}$, it has exponential decay of correlations in the space of Hölder observables, and that the functions $(-t_0, t_0) \ni t \mapsto P_{\text{top}}(f_\alpha, -t \log |Df_\alpha|)$, and $(-t_0, t_0) \ni t \mapsto \mu_{\alpha,t}$ and the Lyapunov exponent

$(-t_0, t_0) \ni t \mapsto \lambda(\mu_{\alpha,t})$ are continuous. Furthermore, it follows from Theorems A to D that there exists a neighborhood $\mathcal{F}$ of f_α and $\mathcal{W}$ of the constant zero potential such that the pressure function $\mathcal{F} \times \mathcal{W} \ni (f, \phi) \mapsto P_{\text{top}}(f, \phi)$ is analytic in ϕ and differentiable in f and that the maximal entropy measure varies differentiably with dynamical system f_α. In particular, the maximal entropy measure varies differentiably with the contact order α of the indifferent fixed point.

Assume now $\alpha > 1$. Then f_α is $C^{2+\beta}$ (with $\beta = 1 - \alpha > 0$) and the potentials $\varphi_{\alpha,t}$ are $C^{1+\beta}$. Moreover, $|\varphi'_{\alpha,t}(x)| \leq |t| 2^\alpha (1+\alpha)\alpha |x|^{\alpha-1}$ can be taken uniformly small, thus satisfying (P'), provided that $|t| \leq t_\alpha$ small. Therefore our results imply that no transition occurs once one considers the order of contact α of the indifferent fixed point to increase. Indeed, there exists a $C^{2+\alpha}$-neighborhood $\mathcal{F}$ of f_α and $C^{1+\alpha}$ neighborhood $\mathcal{W}$ of the constant zero potential such that the pressure function $\mathcal{F} \times \mathcal{W} \ni (f, \phi) \mapsto P_{\text{top}}(f, \phi)$ and the maximal entropy measure function are differentiable on f and analytic on ϕ . As a consequence of Corollary A we also deduce that

$$\begin{array}{ccc} (1, +\infty) \times [-t_\alpha, t_\alpha] & \to & \mathbb{R} \\ (\alpha, t) & \mapsto & P_{\text{top}}(f_\alpha, -t \log |Df_\alpha|) \end{array}$$

is differentiable. Furthermore, the Lyapunov exponent and the Hausdorff dimension of the maximal entropy measure varies continuously with respect to f since

$$\dim_H(\mu_f) = \frac{h_{\mu_f}(f)}{\int \log |Df| \, d\mu_f}.$$

Finally we give some applications to the study of large deviations. These include interesting results for both uniformly and non-uniformly expanding dynamics.

Example 6.3. Assume that f is $C^{1+\alpha}$ local diffeomorphism and $\Lambda \subset M$ be a transitive and f-invariant set such that $f\mid_\Lambda$ uniformly expanding. In [You90], Young obtained a large deviations principle for the unique SRB measure which in our setting generalizes as follows: if ϕ is a Hölder continuous potential then for every $\psi : M \to \mathbb{R}$ continuous

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu_{f,\phi} \left(x \in M : \frac{1}{n} S_n \psi(x) \in [a, b] \right) \leq - \inf_{s \in [a,b]} K_{f,\psi}(s) \tag{6.2}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu_{f,\phi} \left(x \in M : \frac{1}{n} S_n \psi(x) \in (a, b) \right) \geq - \inf_{s \in (a,b)} K_{f,\psi}(s) \tag{6.3}$$

where $K_{f,\psi}(s) = -\sup\big\{-P_{\text{top}}(f,\phi) + h_\eta(f) + \int \phi \, d\eta \colon \int \psi \, d\eta = s\big\}$. We refer the reader to [Va12] for a proof of the previous assertions and extension for weak Gibbs measures. Moreover, if ψ is Hölder continuous then Theorem F yields a large deviation principle where the rate function $K_{f,\phi}$ in (6.2) and (6.2) is replaced by $I_{f,\psi}$, where $I_{f,\psi}(s) = \sup_{-t_\psi \leq t \leq t_\psi} \big\{st - \mathcal{E}_{f,\psi}(t)\big\}$ is the Legendre transform of the free energy function varies differentiably. In particular this proves that the two rate functions above do coincide in the interval $[\mathcal{E}'_{f,\psi}(-t_\psi), \mathcal{E}'_{f,\psi}(t_\psi)]$.

Now, take $T_- = \min\{\int \psi \, d\eta\}$ and $T_+ = \max\{\int \psi \, d\eta\}$ where the minimum and maximum are taken over all f-invariant measures (we omit the dependence on f, ϕ and ψ for notational simplicity). Then for any fixed $t \in [T_-, T_+]$

$$(f, \phi) \mapsto \sup \left\{ P_{\text{top}}(f, \phi) - h_\eta(f) - \int \phi \, d\eta : \eta \in \mathcal{M}_1(f) \text{ and } \int \psi \, d\eta = t \right\}$$

is differentiable, provided that ψ is Hölder continuous. This illustrates the space of invariant probablity measures is rich for uniformly expanding dynamical systems.

In the next examples we study large deviation properties for some open classes of non-uniformly expanding maps.

Example 6.4. Consider the robust class of multidimensional local diffeomorphisms $\mathcal{F}$ obtained by bifurcations of expanding maps as introduced in Example 6.1. It follows from Theorem F that for any Hölder continuous observable ψ not cohomologous to a constant there exists an interval $J \subset \mathbb{R}$ such that for all $f \in \mathcal{F}$ and $[a,b] \subset J$ it follows that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_\phi \left(x \in M : \frac{1}{n} S_n\psi(x) \in [a,b] \right) \leq - \inf_{s\in[a,b]} I_{f,\psi}(s)$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_\phi \left(x \in M : \frac{1}{n} S_n\psi(x) \in (a,b) \right) \geq - \inf_{s\in[a,b]} I_{f,\psi}(s).$$

In particular, a local large deviations principle is obtained, providing in this context a partial answer to Question 3 in [Va12]. In addition, for any injectively parametrized family $V \ni v \to f_v$ the rate function $(s,v) \mapsto I_{f_v,\psi}(s)$ varies continuously with the dynamics and the potential even in this non-uniformly expanding context.

Acknowledgements. The authors are grateful to Welington de Melo for indicating the paper by Franks [Fr79]. This work was partially supported by CNPq-Brazil and FAPESB.

References

[AP06] V. Araújo and M.J. Pacífico. Large deviations for non-uniformly expanding maps. *J. Statist. Phys.*, 125:415–457, 2006.

[AM06] A. Arbieto and C. Matheus. Fast decay of correlations of equilibrium states of open classes of non-uniformly expanding maps and potentials. Preprint www.preprint.impa.br, 2006.

[BY93] V. Baladi and L.S. Young, On the spectra of randomly perturbed expanding maps. *Commun. Math. Phys.*, 156, 355–385, 1993.

[Bal00] V. Baladi. *Positive transfer operators and decay of correlations*. World Scientific Publishing Co. Inc., 2000.

[BG09] V. Baladi and S. S. Gouezel. Good Banach spaces for piecewise hyperbolic maps via interpolation. *Annales de l'Institut Henri Poincaré - Analyse non linéaire*, 26:1453–1481, 2009

[BG10] V. Baladi and S. S. Gouezel. Banach spaces for piecewise cone hyperbolic maps. *J. Modern Dynam.*, 4, 91–137, 2010.

[BS08] V. Baladi and D. Smania. Linear response formula for piecewise expanding unimodal maps. *Nonlinearity*, 21, 677–711, 2008.

[BS09] V. Baladi and D. Smania. Analyticity of the SRB measure for holomorphic families of quadratic-like Collet-Eckmann maps. *Proc. Amer. Math. Soc.* 137, 1431–1437, 2009.

[BS10] V. Baladi and D. Smania. Linear response for smooth deformations of generic nonuniformly hyperbolic unimodal maps. Preprint 2010.

[BT07] V. Baladi and M. Tsujii, Hölder and Sobolev spaces for hyperbolic diffeomorphisms. *Ann.Inst. Fourier*, 57, 27–54, 2007.

[BKL01] M. Blank and G. Keller and C. Liverani. Ruelle-Perron-Frobenius spectrum for Anosov maps. *Nonlinearity*, 15, 1905–1973, 2001.

[Bow75] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lect. Notes in Math.* Springer Verlag, 1975.

[BR75] R. Bowen and D. Ruelle. The ergodic theory of Axiom A flows. *Invent. Math.*, 29:181–202, 1975.

[BL07] O. Butterley and C. Liverani Smooth Anosov ows: correlation spectra and stability. *J. Mod. Dyn.*, 1, 301–322, 2007.

[BF09] J. Buzzi and T. Fisher. Intrinsic ergodicity for certain nonhyperbolic robustly transitive systems. arXiv:0903.3692

[Cas02] A. Castro, Backward inducing and exponential decay of correlations for partially hyperbolic attractors, *Israel J. Math.* 130, 29-75, 2002.

[Cas04] A. Castro, Fast mixing for attractors with mostly contracting central direction, *Ergod. Th. & Dynam. Sys.* 24, 17-44, 2004.

[CV11] A. Castro, P. Varandas. Equilibrium states for non-uniformly expanding maps: decay of correlations and strong stability. *preprint UFBA, 2011*

[DZ98] A. Dembo and O. Zeitouni. Large Deviation Techniques and Applications, Second Edition *Springer Verlag*, 1998.

[DL08] M. Demers and C. Liverani. Stability of statistical properties in two-dimensional piecewise hyperbolic maps. *Trans. Amer. Math. Soc.*, 360, 4777-4814, 2008.

[DPU96] M. Denker, F. Przytycki, and M. Urbański. On the transfer operator for rational functions on the Riemann sphere. *Ergodic Theory Dynam. Systems*, 16:255–266, 1996.

[Dol04] D. Dolgopyat. On differentiability of SRB states for partially hyperbolic systems. *Invent. Math.*, 155, 389–449, 2004.

[El85] R. Ellis. *Entropy, Large Deviation and Statistical Mechanics.* Springer Verlag, 1985.

[Fr79] J. Franks. *Manifolds of C^r Mappings and Applications to Differentiable Dynamical Systems.* Studies in Analysis. Advances in Mathematics Supplementary Studies, vol 4, pages 271–290. Academic Press, 1979.

[GL06] S. Gouezel and C. Liverani. Banach spaces adapted to Anosov systems Ergod. Th. & Dynam. Sys. , 26, 189–217, 2006.

[HM03] T. Hunt and R. MacKay. Anosov parameter values for the triple linkage and a physical system with a uniformly chaotic attractor. *Nonlinearity*, 16:1499–1510, 2003.

[KKPW89] A. Katok and G. Knieper and M. Pollicott M and H. Weiss Differentiability and analyticity of topological entropy for Anosov and geodesic flows. *Inventiones Math.* , 98 581–597, 1989.

[Li95] C. Liverani, Decay of correlations, *Annals of Math.*, 142 (1995), 239–301.

[Lo90] A. Lopes, Entropy and large deviation, *Nonlinearity*, 2, 527–546, 1990.

[OV08] K. Oliveira and M. Viana. Thermodynamical formalism for an open classes of potentials and non-uniformly hyperbolic maps. *Ergod. Th. & Dynam. Sys.*, 28, 2008.

[PSZ08] Ya. Pesin, S. Senti, and K. Zhang. Lifting measures to inducing schemes. *Ergod. Th. & Dynam. Sys.*, 28:553–574, 2008.

[Pin08] V. Pinheiro. Expanding measures. Preprint http://arxiv.org/abs/0811.2545

[PV10] V. Pinheiro and P. Varandas. Thermodynamical formalism for expanding measures. Preprint UFBA.

[PR07] F. Przytycki and J. Rivera-Letelier. Statistical properties of topological Collet-Eckmann maps. *Ann. Sci. Ec. Norm. Sup.*, 40, 135–178, 2007.

[PR10] F. Przytycki and J. Rivera-Letelier. Nice inducing schemes and the thermodynamics of rational maps.. Comm. in Math. Phys. (to appear)

[RY08] L. Rey-Bellet and L.-S. Young. Large deviations in non-uniformly hyperbolic dynamical systems. *Ergod. Th. Dynam. Sys.*, 28: 587–612, 2008.

[Rue76] D. Ruelle. A measure associated with Axiom A attractors. *Amer. J. Math.*, 98:619–654, 1976.

[Rue97] D. Ruelle. Differentiation of SRB states. *Comm. Math. Phys.*, 187, 227–241, 1997.

[Rue04] D. Ruelle. Application of hyperbolic dynamics to physics: some problems and conjectures. *Bull. Amer. Math. Soc. (N.S)* , 41, 275–278 (2004).

[Rue05] D. Ruelle. Differentiating the absolutely contimuous invariant measure of an interval map f with respect to f. *Commun. Math. Phys.* , 258, 445–453 (2005).

[Rue09] D. Ruelle. A review of linear response theory for general differentiable dynamical systems. *Nonlinearity*, 22, 855–870,2009.

[Ru10] H. H. Rugh. Cones and gauges in complex spaces: Spectral gaps and complex Perron-Frobenius theory. *Annals of Math.*, 171: 1707–1752, 2010.

[SV09] M. Sambarino and C. Vásquez. Bowen measure for derived from Anosov diffeomorphims. Preprint arXiv:0904.1036

[Sar99] O. Sarig. Thermodynamic formalism for countable Markov shifts. *Ergodic Theory Dynam. Systems*, 19:1565–1593, 1999.

[Sin72] Ya. Sinai. Gibbs measures in ergodic theory. *Russian Math. Surveys*, 27:21–69, 1972.

[Va08] P. Varandas Correlation decay and recurrence asymptotics for some robust nonuniformly hyperbolic maps. *Journal of Stat. Phys.*, 133:813–839, 2008.

[VV10] P. Varandas and M. Viana, Existence, uniqueness and stability of equilibrium states for non-uniformly expanding maps, *Annales de l'Institut Henri Poincaré- Analyse Non-Linaire*, 27:555–593, 2010.

[Va12] P. Varandas Non-uniform specification and large deviations for weak Gibbs measures. *Journal of Stat. Phys.*, 146, 330–358, 2012.

[Yur03] M. Yuri. Thermodynamical formalism for countable to one Markov systems. *Trans. Amer. Math. Soc.*, 335:2949–2971, 2003.

[You90] L.-S. Young. Some large deviations for dynamical systems. *Trans. Amer. Math. Soc.*, 318:525–543, 1990.

[You98] L.-S. Young. Statistical properties of dynamical systems with some hyperbolicity. *Ann. of Math.*, 147, no. 3, 585-650, 1998.

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